Ridge Approximation and Dimension Reduction for an Acoustic Scattering Model

Paul G. Constantine and Jeffrey M. Hokanson
Department of Computer Science
University of Colorado Boulder
Boulder, Colorado 80309
{paul.constantine,jeffrey.hokanson}@colorado.edu

Drew P. Kouri
Optimization and Uncertainty Quantification
Sandia National Laboratories
Albuquerque, NM 87185
dpkouri@sandia.gov

***Abstract*—Problems in uncertainty quantification (UQ) suffer from the curse of dimensionality. One approach to address this issue is to identify and exploit low-dimensional structure in the underlying model. In this paper, we show that low-dimensional active subspaces are *not* present in a particular model arising from acoustic scattering. This suggests that UQ techniques based on active subspaces are not appropriate for this problem.**

I. Introduction

Uncertainty quantification (UQ) seeks to characterize uncertainty in the output of a complex physical model given a characterization of the uncertainty on the model inputs. Such characterization is challenging when the number of inputs is large due to the *curse of dimensionality*. The only way to bypass the curse of dimensionality is to identify and exploit low-dimensional structure in the map from input parameters to model predictions. Denote a generic deterministic map from inputs to an output quantity of interest (QoI) by $f : \mathbb{R}^M \to \mathbb{R}$. The particular low-dimensional structure that we study, called *ridge structure*, can be loosely described as $f(\boldsymbol{\xi}) \approx g(\boldsymbol{A}^T\boldsymbol{\xi})$ for some matrix $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ with $N < M$ and $g : \mathbb{R}^N \to \mathbb{R}$. This description suggests that the model prediction varies primarily along directions defined by the columns of $\boldsymbol{A}$. When such structure exists in a model, UQ can proceed on the coordinates $\boldsymbol{A}^T\boldsymbol{\xi}$, which is a form of dimension reduction. In this paper, we show numerical evidence that such structure does *not* exist in a UQ problem arising from a Helmholtz-based model for acoustic scattering with a random wave number. The absence of this structure is interesting given how prevalent such structures are in physics-based models [1, Chapter 5].

We pursue two numerical techniques for assessing the presence of ridge structure. The first is based on *active subspaces* [1], which are eigenspaces of the matrix

$$\boldsymbol{C} = \mathbb{E}\left[\nabla f(\boldsymbol{\xi})\,\nabla f(\boldsymbol{\xi})^T\right], \tag{1}$$

where the expectation is with respect to the given measure on $\boldsymbol{\xi}$. The null space of $\boldsymbol{C}$ identifies directions along which f is constant. Rapid decay in eigenvalues reveals ridge structure, where the important directions are the associated eigenvectors. When gradients are available as a subroutine, we can approximate $\boldsymbol{C}$ and its eigenpairs with simple Monte Carlo. The second approach we use to assess ridge structure is discrete least-squares. Let $(\boldsymbol{\xi}_i, f(\boldsymbol{\xi}_i))$ be given input/output pairs, and consider the nonlinear least-squares problem,

$$\underset{\boldsymbol{U}\in\mathbb{G},\, g\in\mathbb{P}_k}{\text{minimize}} \; \sum_{i=1}^{n} \left(f(\boldsymbol{\xi}_i) - g(\boldsymbol{U}^T\boldsymbol{\xi}_i)\right)^2, \tag{2}$$

where $\mathbb{G}$ is the Grassmann manifold of N-dimensional subspaces, and $\mathbb{P}_k$ is the set of polynomials of total degree k in N variables. We call this approach *ridge approximation* [2].

II. A Helmholtz Model for Acoustic Scattering

The model is motivated by direct field acoustic testing (DFAT) [3]. DFAT is a nondestructive method for testing components of engineered systems. The typical setup for DFAT experiments is: (i) surround the component with speakers; (ii) bombard the component with acoustic pressure; and (iii) measure the vibrational response with, e.g., accelerometers. The accelerometer data can then be used to infer material properties of the component. An important QoI in DFAT is the deviation of the acoustic pressure field inside the component from a target pressure field. We model the acoustic pressure field as the solution to Helmholtz equation with random refractive index. See [4] for DFAT optimal control problems constrained by Helmholtz equation.

We consider an idealized DFAT example in the physical domain $D = (-5, 5)^2$. Let $D_R = \{x \in D : \|x\|_2 \le 2\}$ define the component subdomain and $\bar{w}(x)$ denote the target pressure profile. Let $D_C = \{x \in D : 2.5 \le \|x\|_2 \le 2.6\}$ denote the speaker subdomain. Fig. 1 depicts the experimental setup. The QoI is

$$J(u) = \frac{1}{2}\int_{D_R} (u(x) - \bar{w}(x))\overline{(u(x) - \bar{w}(x))}\mathrm{d}x, \tag{3}$$

where u is an acoustic pressure field. The uncertain wave number is $k(x) = 10$ if $x \notin D_R$ and $k(x) = 10\,(1 + \varepsilon(x))$ if $x \in D_R$, where ε is the uncertain refractive index.

This work was supported by the Department of Defense, Defense Advanced Research Project Agency's program Enabling Quantification of Uncertainty in Physical Systems. Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the US Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

We model ε by the truncated Karhunen-Loève expansion $\varepsilon(x) = \sum_{m=1}^{M} \phi_m(x)\,\xi_m$, with $M = 40$ terms. Here, $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_m]^T$ is a vector of uncorrelated Gaussian random variables with mean zero and unit variance. This model for refractive index has zero mean, and the basis $\{\phi_m\}$ is derived from eigenpairs of the squared-exponential covariance kernel $0.01^2 \exp(-(x-x')^2/L)$ with correlation length $L > 0$. The acoustic source $s : D \to \mathbb{C}$ is $(1+i)$ in D_C and 0 elsewhere. The acoustic pressure field $u = u(x)$ solves the Helmholtz equation:

$$-\Delta u - k^2 u = s \;\text{ in } D, \qquad \frac{\partial u}{\partial n} = i k_0 u \;\text{ on } \partial D. \qquad (4)$$

The real and imaginary components of the desired wave pressure $\bar{w}(x)$ are plotted in Fig. 1 with $k_0 = 10$. We discretize

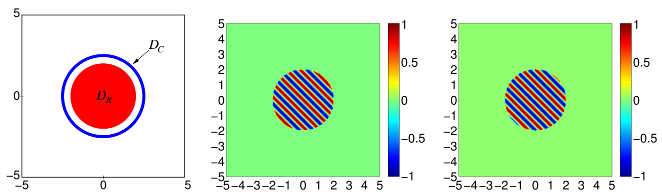

Fig. 1. Speaker subdomain, D_C, and component subdomain, D_R (left). Real (center) and imaginary (right) parts of the desired state, $\bar{w}$.

(4) in space using continuous Q1 finite elements on a uniform mesh of 200×200 quadrilaterals.

III. Numerical Results

The first test we apply to the DFAT QoI (3) as a function of the 40 refractive index coefficients $\boldsymbol{\xi}$ is to randomly sweep the parameter space along random directions to assess smoothness. Fig. 2 shows three representative sweeps revealing that the QoI is smooth but oscillatory. Fig. 3 shows the first 11 eigenvalues of a 20 000-sample estimate of $\boldsymbol{C}$ from (1) and bootstrap-based estimates of the subspace errors. The order-of-magnitude gap between the first and second eigenvalues suggests 1-d ridge structure. Fig. 4 shows a shadow plot (right) of the QoI versus one linear combination of $\boldsymbol{\xi}$ computed as in (2) with a 10 000-sample subset. The leftmost plot shows the 40 components of the computed direction with 10 different data subsets; the plot suggests that computation of these weights is stable. The shadow plot shows that the QoI varies in more than one direction, since the plotted data does *not* look like it comes from a univariate function.

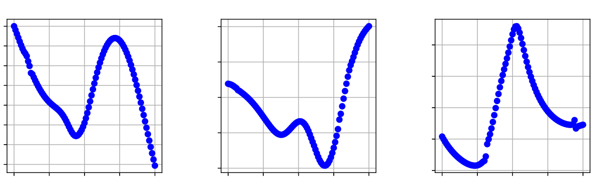

Fig. 2. Random 1d sweeps of the QoI.

Table I shows the 2-norm error in a 10 000-sample testing set for a variety of polynomial degrees and subspace dimensions in (2). The slow decrease in error as degree increases

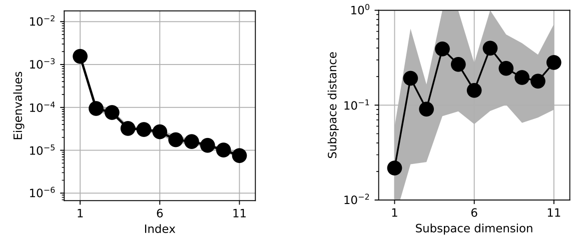

Fig. 3. First 11 eigenvalues (left) of $\boldsymbol{C}$ from (1) and bootstrap-based estimates of Monte Carlo subspace errors (right).

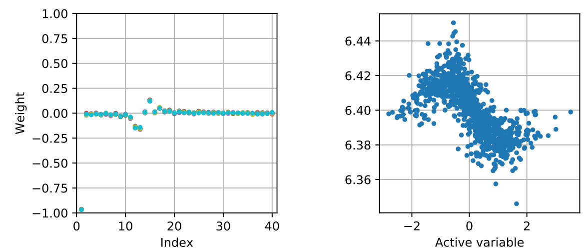

Fig. 4. One-d shadow plot (right) and associated weights (left).

TABLE I
Testing Error in Ridge Approximation (2).

deg \ dim	1	2	3	4	5
2	1.14e+0	1.14e+0	1.13e+0	1.13e+0	1.13e+0
3	9.89e-1	9.63e-1	9.56e-1	9.52e-1	9.50e-1
4	9.72e-1	9.48e-1	9.28e-1	9.25e-1	9.17e-1
5	9.12e-1	8.73e-1	8.52e-1	8.66e-1	8.53e-1

and dimension increases provides further evidence that low-dimensional ridge structure is *not* present in the QoI as a function of the 40 input parameters.

IV. Conclusion

We have determined that a particular QoI from a particular acoustic scattering model as a function of 40 coefficients parameterizing the refractive index does *not* contain low-dimensional ridge structure. This result may help decide among available methods for quantifying uncertainty. One might expect that the responses in related scattering problems might exhibit similar, intrinsically high-dimensional structure.

References

[1] P. G. Constantine, *Active Subspaces: Emerging Ideas in Dimension Reduction for Parameter Studies*. SIAM, Philadelphia, 2015. [Online]. Available: https://doi.org/10.1137/1.9781611973860

[2] P. G. Constantine, A. Eftekhari, J. Hokanson, and R. A. Ward, "A near-stationary subspace for ridge approximation," *Computer Methods in Applied Mechanics and Engineering*, vol. 326, pp. 402–421, 2017. [Online]. Available: https://doi.org/10.1016/j.cma.2017.07.038

[3] P. Larkin and M. Whalen, "Direct, near field acoustic testing," SAE, Tech. Rep., 1999, technical Paper 1999–01–5553.

[4] D. P. Kouri, M. Heinkenschloss, D. Ridzal, and B. G. van Bloemen Waanders, "A trust-region algorithm with adaptive stochastic collocation for PDE optimization under uncertainty," *SIAM Journal on Scientific Computing*, vol. 35, no. 4, pp. A1847–A1879, 2013. [Online]. Available: https://doi.org/10.1137/120892362