Issued: 10 March 2020
Due: 17 March 2020

Graduate Students: one-on-one meetings about projects will take place during the
week of 17 March. I will send around a scheduling doodle this week; make sure you
sign up for a slot.

Reading: Liz’s notes on Wolf’s algorithm and the materials listed in the “Assigned
reading for PS8-10” handout on the course webpage.

Online assignment: Tuesday: unit 7.2 video. Thursday: unit 9.2 videos. Friday:
quizzes 7.2 and 9.2.

Bibliography:

- H. Abarbanel et al., “Lyapunov exponents in chaotic systems: their importance
  and their evaluation using observed data,” *International Journal of Modern Physics

- H. Abarbanel et al., “Variation of Lyapunov Exponents on a Strange Attractor,”

- G. Gunaratne et al., “Chaos beyond Onset: A Comparison of Theory and Ex-
  unstable periodic orbits (UPOs).

- Predrag Cvitanovic’s publications page\(^1\) has lots of good papers on UPOs, including
  61:2729-2732 (1988), the one that defines attractors as made up of UPOs.

- J.-P. Eckmann and D. Ruelle, “Ergodic theory of chaos and strange attractors,”
  from time series,” *Phys. Rev. A* 34:4971 (1986). The first one proposed the
  “reverse engineer the Jacobian” approach to $\lambda$ calculation; the second applied it
to real data.

- U. Parlitz, “Identification of True and Spurious Lyapunov Exponents from Time

\(^1\)http://www.cns.gatech.edu/ predrag/papers/preprints.html


Problems:

1. (a) Generate a trajectory from the Lorenz system with $a = 16$, $r = 45$, $b = 4$. Use nonadaptive RK4 with a timestep of 0.001 from some starting point near the attractor, and generate at least 15000 points.

(b) Generate a non-chaotic trajectory by lowering the $r$ value to 20 or less and repeating the run.

The next two problems will use these trajectories as “experimental” chaotic and non-chaotic data sets. (There is no need to turn anything in for problem 1.)

2. Implement the Wolf algorithm for determining $\lambda_i$ of an experimental data set and test it on the trajectories from problem 1. The notes listed in the assigned reading include a schematic that should help you think about the algorithm. Note that you do not have to worry about embedding here, as the data are already in full state-space form. To run the Wolf algorithm on a data set that only contained samples of one state variable, you’d first have to embed that data, using your PS8 results and making intelligent choices for $m$ and $\tau$. (CSCI 5446 students will do that in the next problem.)

Are your $\lambda_i$ results on these two data sets consistent with what you know about Lyapunov exponents and chaos? Why or why not?

3. [optional for those enrolled in CSCI 4446] Save out just the $x$ coordinate of the data set from problem 1(a) to a file (say, foo) and use TISEAN’s lyap_k command to compute $\lambda_1$ from that scalar time series. On the CSEL machines, the command will look like this:

   lyap_k foo -m? -M?? -d??? -o outputfile

...where you want the calculation done for a range of values from ? to ?? of the dimension $m$ and ??? is the delay $\tau$. That is, if I wanted lyap_k to run with $\tau = 20$ and $1 \leq m \leq 4$, I would type lyap_k foo -m1 -M4 -d20 -o outputfile. (If you are using the Mac version of TISEAN, the command looks like this instead: lyap_k foo -M1,4 -d20 -o outputfile.)

Use mutual on each data set, as you did in PS8, to pick $\tau$. Use the Takens theorem to pick $m$ (this works here because we know the true dimension; in general, that’s not the case). Please explain your choices for these two parameters, and also compare your answer for $\lambda_1$ to the value that you obtained for this trajectory in the previous problem, using Wolf’s algorithm. Discuss any discrepancies.

[totally optional for everyone, but useful for those who are interesting in actually using these techniques] Play with at least one of the other parameters of the lyap_k algorithm: the Theiler window -t, the epsilon size -r (and, if you wish, the number of iterations
Describe and explain their effects upon the results. These explanations are going to require some digging around on the TISEAN website and corroboration against the class readings and your lecture notes. Please comment on which of the results you trust more and explain why. What were the “lessons learned” from your experience with this part of this problem?

4. Given the system derivative, the variational equation (PS7), and a linear algebra package that computes eigenvalues$^2$, calculating Lyapunov exponents is very easy:

$$\lambda_i := \lim_{t \to \infty} \frac{1}{t} \ln |m_i(t)|$$

where $m_i(t)$ are the eigenvalues of the matrix $\{\delta_{ij}\}$ that you integrated in PS7. Of course, you can’t let $t$ go to $\infty$ in practice; even letting it get very big can cause numerical issues, as we’ve discussed in class (viz., long, skinny variations, which will make $\{\delta_{ij}\}$ be ill-conditioned). But you don’t want it too small, either, or you won’t be capturing the asymptotic behavior of the system—which is what Lyapunov exponents are about. In part (a) below, I suggest a value for $t$; in general, though, one has to do some thoughtful experimentation to find a good time period over which to do the calculation here. (Part (b) of this question asks the CSCI 5446 people to do a bit of thinking about that.)

(a) Calculate the $\lambda_i$s for the Lorenz system, using the same initial conditions and parameter values that you used in problem 1(a) above. Approximate the “$t \to \infty$” step in the equation above by using a 10000-point integration run. Comment on your results: whether or not they match the values from Problem 3, whether or not they should match those values, which ones you should trust more and why, etc.

(b) [optional for those enrolled in CSCI 4446] Repeat the calculation for longer and shorter variational equation integration runs and comment on the results. (Hint #1: do the $\lambda$s change? Should they, theoretically? Hint #2: watch out for ill-conditioned matrices here...) Next, try moving the initial condition (to a different point on the attractor and seeing what that does to your results.

Please include a sentence in your writeup about those experiments: your choices and your reasoning about them.

$^2$Eigenvalues and eigenvals in Mathematica and Maple, respectively.