Reading: Strogatz, sections 6.0-6.5; Liz’s notes on the variational equation; Parker&Chua, Appendix B.

Online assignment: Tuesday: unit 3.5 and 6.4 videos. Thursday: unit 6.5 video. Friday: quiz 6.4.

Bibliography:

- D. Ruelle, “Strange Attractors,” The Mathematical Intelligencer, 2:126-137, 1980. There’s a closely related paper by him in the Cvitanovic collection, which is on library reserve.

Problems:

1. [math] Derive the Jacobian $D\vec{F}$ for the Lorenz system:

\[
\vec{F}(\vec{x}, a, r, b) = \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
a(y - x) \\
rx - y - xz \\
xy - bz
\end{bmatrix}
\]  

(1)

2. [math] Write down the associated variational system $\dot{\delta} = D\vec{F}\delta$. The product of the Jacobian matrix $D\vec{F}$ and the $n \times n$ matrix of variations

\[
\delta = \begin{bmatrix}
\delta_{xx} & \delta_{yx} & \delta_{zx} \\
\delta_{xy} & \delta_{yy} & \delta_{zy} \\
\delta_{xz} & \delta_{yz} & \delta_{zz}
\end{bmatrix}
\]
yields an $n \times n$ matrix of the derivatives of the variations $\dot{\delta}$. $\delta_{xy}$, for instance, is the component of the $x$-variation that comes from the previous $y$-variation, and $\dot{\delta}_{xy}$ is its rate of change with time.

3. [programming] A combination of this variational derivative and the original system derivative from PS5 can be used to integrate the $(n^2 + n)$-dimensional variational equation

$$\begin{cases}
\dot{x} \\
\dot{\delta}
\end{cases} = \begin{cases}
\vec{F} \\
D_x \vec{F} \cdot \delta
\end{cases}$$

from the initial condition

$$\begin{cases}
\vec{x}_0 \\
I
\end{cases}$$

with $t = t_0$. The time evolution of the first $n$ elements of this set of initial conditions follows the trajectory $\phi_t(\vec{x}_0, t_0)$. The row and column sums of the matrix formed by the next $n^2$ elements are different ways to look at the evolved versions of the initial variations: the first column sum, for example, gives the $x$ component of the evolved variation, while the first row sum tells you what the $x$-piece of the original variation has grown into.

Integrate the Lorenz variational equation using RK4 (not ARK4) from the following initial conditions for 100 steps. Use $a = 16$, $r = 45$, $b = 4$, and a timestep of .001. In each case, give

- the components of the evolved matrix $\delta$
- the evolved variations (the column sums of $\delta$)

at the endpoint of the trajectory. Use $t_0 = 0$ (this is really immaterial, as the system is autonomous). You need only turn in these twelve numbers for each of the following questions; no plots or intermediate values are necessary.

(a) $\begin{bmatrix} x & y & z & \delta_{xx} & \delta_{xy} & \delta_{xz} & \delta_{yx} & \delta_{yy} & \delta_{yz} & \delta_{zx} & \delta_{zy} & \delta_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & -5 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

(d) Look carefully at the evolved matrices of variations and describe some of their interesting features. From which point [(a), (b), or (c)] do the variations grow fastest? In which direction $(x, y, z)$? Do you notice any symmetries or gross differences between the different points (e.g., “the $y$-variation grows really fast near point A, but less so near point B...“)?