University of Colorado
Department of Computer Science

Chaotic Dynamics – CSCI 4446/5446
Spring 2019

Problem Set 7

Issued:	26 February 2019
Due:	5 March 2019

Reading: *Strogatz, sections 6.0-6.5; Liz's notes on the variational equation*; Parker&Chua, Appendix B.

Online assignment: Tuesday: unit 3.5 and 6.4 videos. Thursday: unit 6.5 video. Friday: quiz 6.4.

Bibliography:

- H.-D. Chiang *et al*, "Stability Regions of Nonlinear Autonomous Dynamical Systems," *Trans. Auto. Control* **33**:1, 1988.
- R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin/Cummings, 1986.
- C. Grebogi, E. Ott, and J. A. Yorke, "Chaos, Strange Attractors and Fractal Basin Boundaries in Nonlinear Dynamics," *Science* **238**:632-638, 1987.
- D. Ruelle, "Strange Attractors," *The Mathematical Intelligencer*, **2**:126-137, 1980. There's a closely related paper by him in the Cvitanovic collection, which is on library reserve.

Problems:

1. *[math]* Derive the Jacobian $D_{\vec{x}}\vec{F}$ for the Lorenz system:

$$\vec{F}(\vec{x}, a, r, b) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix} \tag{1}$$

2. *[math]* Write down the associated variational system $\dot{\delta} = D_{\vec{x}}\vec{F}\delta$. The product of the Jacobian matrix $D_{\vec{x}}\vec{F}$ and the $n \times n$ matrix of variations

$$\delta = \begin{bmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{bmatrix}$$

yields an $n \times n$ matrix of the derivatives of the variations $\dot{\delta}$. δ_{xy}, for instance, is the component of the x-variation that comes from the previous y-variation, and $\dot{\delta}_{xy}$ is its rate of change with time.

3. *[programming]* A combination of this variational derivative and the original system derivative from PS5 can be used to integrate the (n^2+n)-dimensional variational equation

$$\left\{ \begin{array}{c} \dot{\vec{x}} \\ \dot{\delta} \end{array} \right\} = \left\{ \begin{array}{c} \vec{F} \\ D_{\vec{x}}\vec{F} \cdot \delta \end{array} \right\}$$

from the initial condition

$$\left\{ \begin{array}{c} \vec{x}_0 \\ I \end{array} \right\}$$

with $t = t_0$. The time evolution of the first n elements of this set of initial conditions follows the trajectory $\phi_t(\vec{x}_0, t_0)$. The row and column sums of the matrix formed by the next n^2 elements are different ways to look at the evolved versions of the initial variations: the first column sum, for example, gives the x component of the evolved variation, while the first *row* sum tells you what the x-piece of the *original* variation has grown into.

Integrate the Lorenz variational equation using RK4 (not ARK4) from the following initial conditions for 100 steps. Use $a = 16$, $r = 45$, $b = 4$, and a timestep of .001. In each case, give

- the components of the evolved matrix δ
- the evolved variations (the column sums of δ)

at the endpoint of the trajectory. Use $t_0 = 0$ (this is really immaterial, as the system is autonomous). You need only turn in these twelve numbers for each of the following questions; no plots or intermediate values are necessary.

(a) $[x\ y\ z\ \delta_{xx}\ \delta_{xy}\ \delta_{xz}\ \delta_{yx}\ \delta_{yy}\ \delta_{yz}\ \delta_{zx}\ \delta_{zy}\ \delta_{zz}] = [0\ 1\ 2\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1]$

(b) [10 -5 2 1 0 0 0 1 0 0 0 1]

(c) [0 -1 2 1 0 0 0 1 0 0 0 1]

(d) Look carefully at the evolved matrices of variations and describe some of their interesting features. From which point [(a), (b), or (c)] do the variations grow fastest? In which direction (x, y, z)? Do you notice any symmetries or gross differences between the different points (e.g., "the y-variation grows really fast near point A, but less so near point B...")?