University of Colorado
Department of Computer Science

Chaotic Dynamics – CSCI 4446/5446
Spring 2019

Problem Set 11

Issued:	2 April 2019
Due:	9 April 2019

Note: only undergraduates are required to turn in this problem set, but *all* students should do the reading listed below.

Reading: *Liz's Classical Mechanics notes and pp428-429 of Strogatz (This is a short section called "The Importance of Dissipation." The page numbers may be different in your edition. It's right before section 12.2.).*

Online assignment: Tuesday: unit 8.6 video. Thursday: unit 9.4 video. No quizzes this week.

Bibliography:

- R. Aitken, *The Binary Stars*, McGraw-Hill, 1935 (and Dover reprint 1964)
- V. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, second edition, 1989.
- E. Bradley and R. Mantilla, "Recurrence Plots and Unstable Periodic Orbits," *Chaos* **12**:596-600 (2002). On my publications webpage.
- J. Danby, *Fundamentals of Celestial Mechanics*, Willmann-Bell, second edition, 1992. On library reserve.
- H. Goldstein, *Classical Mechanics*, Addison-Wesley, second edition, 1980.
- J.-P. Eckmann, S. Kamphorst, and D. Ruelle, "Recurrence plots of dynamical systems," *Europhysics Letters* **4**:973–977 (1987).
- J. Iwanski and E. Bradley, "Recurrence Plot Analysis: To Embed or not to Embed?," *Chaos*, **8**:861-871 (1998). On my publications webpage.
- I. Peterson, *Newton's Clock: Chaos in the Solar System*, Freeman, 1993. On library reserve.
- H. Poincaré, "Les Méthodes Nouvelles de la Méchanique Céleste," Gauthier-Villars, Paris, 1892.

- L. Taff, *Celestial Mechanics: A Computational Guide for the Practitioner*, Wiley, 1985.

Problems:

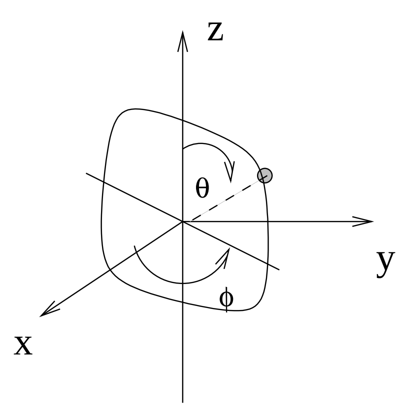
1. For this problem, you'll need a bike wheel, knowledge of section 3 of the *Classical Mechanics Notes*, and a little bit of artistic skill. Hold each end of the axle (or the skewer, if you're fastidious about grease on your hands) with one hand, straight out in front of you, keeping your hands at the same level and your palms down, so the spokes are vertical. Spin it so that the top is approaching you.

(a) Sketch yourself holding the wheel. Show $\vec{\omega}$ and $\vec{L}$ on your sketch.

Now try to turn the wheel sideways, so your left hand is directly above your right hand.

(b) Which way does the wheel pull you as it tries to preserve the angular momentum $\vec{L}$? Does this make sense? Why?

2. (a) Write down the Lagrangian for a bead of mass m moving freely on a rotating hoop, as shown below:



Assume that the hoop is a perfect circle of radius r, that it can rotate freely around the vertical (z) axis passing through its center, and that the bead moves freely along this circle. This system has two degrees of freedom, so you'll need two coordinates. For this problem, please use the angle θ between the bead and the vertical axis as one coordinate and the angle ϕ between the hoop and the x axis as the other.

(b) Use the Lagrangian to derive (not "solve," not "integrate," just "derive") the equations of motion.

(c) Given a Hamiltonian or Lagrangian, how do you identify *cyclic coordinates*?

(d) Are there any conserved quantities ("constants of the motion") in this system? If so, which one(s)?

3. Your task in this problem is to learn how to set up the two-body problem on a binary star system. Begin by playing with this applet and get a feel for the effects of different initial conditions and masses:

`phet.colorado.edu/sims/my-solar-system/my-solar-system_en.html`

For the rest of this assignment, you'll need to refer to section 4 of the *Classical Mechanics Notes* for information, equations, and derivations.

If you were observing a binary star system from some space-fixed origin point, you would need 12 pieces of information to describe the positions and velocities of both stars, and you'd use the 12^{th}-order two-body equations — equation (4) in the *Notes* — to simulate their behavior.

If, on the other hand, you were sitting on one member of a binary star system, with the other star orbiting around you[1], the 12^{th}-order two-body system — equation (4) in the *Notes* — reduce to a 6^{th}-order system (equation (5)). The missing six pieces of information describe the position and velocity of the star that you're sitting on.

Make sure you understand the derivation of the equations of motion in both frames, and try to visualize the various ellipses involved. Each star is on an ellipse around the other *and* around the center of mass. (The "around the center of mass" part of this is very hard to visualize, partially because the center of mass moves.)

To fill in the various initial conditions in the various two-body equations, we will use conservation of angular momentum and some simple properties of ellipses. Note: M is the total mass $m_1 + m_2$, μ is the reduced mass $\frac{m_1 m_2}{M}$, and $\gamma = GM$.

Conservation of angular momentum tells you that the motion is planar: the two stars continue to revolve around each other in the plane in which they start out. This takes care of four of the twelve initial conditions (all of the "out-of-plane" position and velocity components).

To make things simple, assume that the stars are of equal, normalized masses ($m_1 = m_2 = 0.5$), and set both the semimajor axis a of the orbit and the gravitational constant G to 1.0. These simplifications are not as wild as they might seem; they are drawn from a classic *Astrophysical Journal* article by Hut and Bahcall.

(a) Sketch and dimension the elliptical orbit of one star around the other (pick either one) for an eccentricity e of 0.7. You can try the string-and-two-thumbtack trick: make a loop of string, thread it around two thumbtacks (pinning down the paper at the foci), put a pencil inside it, stretch out as far as the string will let you, and draw a closed curve, keeping the string loop taut.

Can you sketch the ellipse of your star's orbit around the other — on the same plot?

(b) Repeat (a) for $e = 0$. This is what Hut and Bahcall used for the binary in the

[1] That is, you're at one of the foci of the ellipse that it's following

complicated binary-field star exchange interaction that I have showed in class several times.

(c) On your sketches from parts (a) and (b), show (i) the periapses and (ii) the positions of the stars when the true anomalies in each orbit are $\frac{\pi}{4}$.

4. The final task in setting up the two-body equations is to choose a coordinate system and figure out the initial conditions in that frame of reference. Common choices are to place the origin at the center of mass or on one of the masses involved; in addition to this, one must specify whether the coordinate system is fixed in space or moves with some associated point (one of the stars or the center of mass of the pair of stars). For this problem, please use a *space-fixed* coordinate system and place its origin at the initial position of one member of the binary. Remember that this set of axes will remain at the same point in the same orientation while the star that was originally at the origin moves away along its orbit.

Assume that the star at the origin is at rest, and that the binary orbits — with eccentricity $e = 0$ — in the $x - y$ plane, with the periapse of the second star on the positive x-axis. Recall that the period and semimajor axis of an elliptical orbit are related via Kepler's third law: $P^2 = \frac{4\pi^2 a^3}{\gamma}$. You are now equipped to figure out all of the initial conditions for a RK4 run on either set of two-body equations.

(a) Compute the period of the orbit using Kepler's third law.

(b) Draw a picture of the coordinate axes and show the positions of both stars at $t = 0$.

(c) Compute the velocity of the star that is not at the origin. Indicate its direction on your drawing from part (b).

(d) Use these results to fill in the twelve components of $\vec{r_1}$, $\vec{r_2}$, $\dot{\vec{r_1}}$, and $\dot{\vec{r_2}}$ for the two-body equation. (Hint: you've already done this, implicitly, in constructing the previous drawing — all you have to do here is write down the components in the right places.)

(e) Compute the six components of $\vec{r}$ and $\dot{\vec{r}}$ for the reduced two-body equation for this binary system. Which star, in your drawing from part (b), did you choose as the origin?

Problem set 12 will cover writing the equations out and actually running this integration.