Classical Mechanics
Notes for CSCI4446/5446

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1 Introduction

Classical mechanics is the study of the motion of "material bodies"[3]. First-semester college physics (force, work, energy, momentum, torque, tension, pulleys, masses on springs, etc.) lays the groundwork for this, but the field is far richer, and physics majors take at least one more full semester of classes on this topic. These notes give an extremely brief introduction to three of the most interesting topic areas that are treated in such a course:

- Lagrangians and Hamiltonians

  Lagrangians and Hamiltonians are different (and very useful) ways to write $F = ma$ that let you avoid having to deal with vector-valued force balances. They not only make hideous mechanics problems easy, but they also expose deep symmetries and conserved properties.

- Rigid-body dynamics

  Rigid-body dynamics is the study of how objects like baseballs, planets, tops, and snowflakes move through space.

- Gravitation

  Gravitation concerns the intricacies of the n-body problem: n masses pulling on one another in the standard GmM/r^2 way. For three or more bodies, this can get complicated and interesting.

At the end of this section of the course, we will put these building blocks together to understand how celestial bodies interact with one another and move through space...often chaotically.

These notes only scratch the surface of this field; I describe methods and concepts using examples, without doing any of the derivation or justification. There are dozens of textbooks available if anyone is interested in probing further into this material. A few of the best, from a dynamical systems point of view, are Arnol'd[1] and Goldstein[3], both of which are on reserve for this course; the former is somewhat heavier going than the latter.

2 Lagrangians and Hamiltonians

Lagrangians and Hamiltonians are mathematical techniques for solving mechanics problems — that is, given a system, they help you find the ODE that governs its behavior. The word "Hamiltonian," as an adjective, has another meaning: that a system does not gain or lose energy over the course of time from/to processes like friction. "Hamiltonian" is synonymous with "conservative" or "non-dissipative." In these notes, I will only address conservative systems.

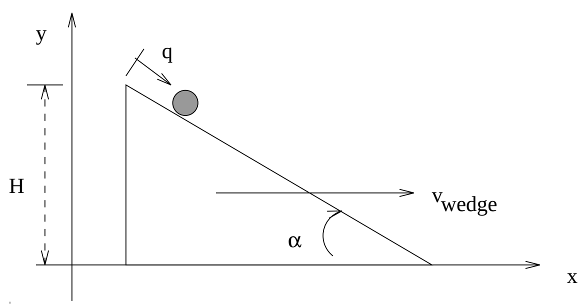
2.1 Lagrangians

To derive the equations of motion of a conservative system using the Lagrangian, you follow a seven-step procedure:

1. figure out how many degrees of freedom the system has
2. choose one "generalized coordinate" for each
3. write down the potential energy, V, in terms of those coordinates
4. write down the kinetic energy, T, in terms of those coordinates
5. write down the Lagrangian: $\mathcal{L} = T - V$
6. take a bunch of derivatives
7. plug those derivatives into Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{1}$$

I'll go through an example that shows these steps. Consider a ball on a wedge-shaped block that is sliding at constant velocity across a plane:

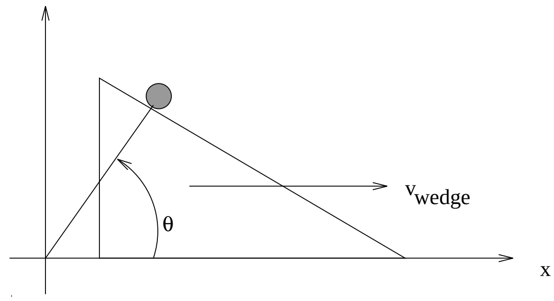


At first glance, it may seem that this problem has two degrees of freedom: that you'd need to write down a force balance in x and another one in y to derive the *equations of motion* that describe the position of the ball, and that the solutions to those equations would have four pieces: $x(t)$, $\dot{x}(t)$, $y(t)$, and $\dot{y}(t)$[1]. However, there's an extra constraint in the problem: the "normal" force that keeps the ball on the surface of the wedge (and not inside it). Because of this, the "ball on a sliding wedge" system in the picture above really has only one *degree of freedom*. The constraint force between ball and surface has reduced the "size" of the problem.

The second task in the procedure is to pick a coordinate that "measures" the system's state in the physical dimension parametrized by that degree of freedom. Here, one obvious[2] choice is the distance from ball to the top of the wedge (q, in the previous figure). Good choices of coordinates are important to the success of the procedure; if you pick them well, the math will be easy and the symmetries obvious. If you choose an "unnatural" coordinate — say, the angle θ between the x-axis and the line from the ball to the origin:

[1] Recall that the phase or state space has two axes — position and velocity — for each degree of freedom, and the number of axes is called the *dimension* of the system.

[2] Good coordinate choice isn't initially — or always — "obvious;" doing it well is a matter of practice.



...the math will be hard and the symmetries invisible.

The third and fourth tasks involve writing the potential and kinetic energies in terms of these coordinates, and this is where the requirement that the system is conservative kicks in. In a nonconservative (or dissipative) system, the word "potential" doesn't even make sense: if the amount of work you do to move from one place to another in the field depends on the path taken (e.g., if there's friction), then you simply can't talk about potential or potential energy.

Higher-level tangent:

The formal way to test whether a system is conservative is to take the curl or path integral of the force field; from the dim recesses of your vector calculus memory, you may recall a formula like:

$$\nabla \times \vec{F} = \oint \vec{F} \cdot \vec{dr}$$

If this whole mess is zero, one does the same amount of work going from point A to point B, independent of the path taken, and the field is conservative.

Liouville's Theorem: the "phase flow" of a conservative system — how the system evolves under the influence of a conservative field[3], as plotted in phase space — preserves volumes. That is, a unit-volume cube of initial conditions will deform over time (as you investigated in PS7), but its volume will remain constant.

T is generally the easier of the two energies to write down; it usually only involves figuring out how to express $\frac{1}{2}mv^2$ in terms of the generalized coordinates q_i. In the sliding wedge example, $x = v_{wedge}t + q\cos\alpha$ and $y = H - q\sin\alpha$. Since $v^2 = \dot{x}^2 + \dot{y}^2$,

$$T = \frac{1}{2}m\left[\dot{q}^2\cos^2\alpha + 2\dot{q}v_{wedge}\cos\alpha + v_{wedge}^2 + \dot{q}^2\sin^2\alpha\right]$$

In simple mechanics problems, V is usually just mgh — again, written in terms of the generalized coordinates. The only difficult part here is getting the potential energy to be zero at the "lowest" state. For the wedge, $V = mg(H - q\sin\alpha)$, so the Lagrangian is:

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{q}^2 + 2\dot{q}V\cos\alpha + V^2) - mg(H - q\sin\alpha)$$

Note the combination of the $\dot{q}^2\sin^2\alpha$ and $\dot{q}^2\cos^2\alpha$ terms into $\dot{q}^2$.

Lagrange's equations require various partial and total derivatives of $\mathcal{L}$. The only hard part of this is remembering what's a function of time (e.g., $q(t)$ and $\dot{q}(t)$) and what's constant (e.g., V, α) and taking the derivatives appropriately; the former play different roles when you're doing a $\frac{d}{dt}$ derivative and a $\frac{\partial}{\partial t}$ derivative. In the sliding wedge problem,

$$\frac{\partial\mathcal{L}}{\partial\dot{q}} = m\dot{q} + mv_{wedge}\cos\alpha$$

[3]Here, "field" can mean the physical force field or the ODE that models it.

$$\frac{\partial \mathcal{L}}{\partial q} = mg \sin \alpha$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = m\ddot{q}$$

Plugging these derivatives into Lagrange's equations yields

$$m\ddot{q} - mg \sin \alpha = 0$$

This 2^{nd}-order ODE is the equation of motion for the system. If you pull it apart into two first-order ODEs (cf., PS3 problem 2), you get something[4] like:

$$\begin{aligned} \dot{q} &= \zeta \\ \dot{\zeta} &= g \sin \alpha \end{aligned}$$

If you chuck these equations into your RK4 code (PS4 or PS5), the $q(t)$ and $\zeta(t) = \dot{q}(t)$ values that you get out are the coordinates of points on the state-space trajectory of the system. Note that the equation is pretty simple in retrospect — it just looks like the ball is "falling" under the influence of a reduced gravity — but that simplicity probably wasn't apparent to you from your initial examination of the problem.

The sliding block example has a single degree of freedom and thus a single coordinate. An n-degree of freedom system requires you to choose n generalized coordinates q_i and the seven-step Lagrangian process requires you to take $3n$ derivatives and yields n 2^{nd}-order ODEs. The next example, the spherical pendulum, will demonstrate how the whole procedure works for a two degree-of-freedom problem.

Consider a ball of mass m on the end of a massless rigid rod of length l that is attached to a universal joint (something that lets the rod swing in all directions, not just in a plane). The natural coordinates to choose in this case are some sort of off-vertical angle θ — measured either from hanging straight down or from standing straight up — and an azimuthal angle ϕ. The latter is like longitude and former is like either latitude or co-latitude, depending on where $\theta = 0$ is defined. The rest of this problem assumes $\theta = 0$ when the pendulum is at its inverted point and $\theta = \pi$ when it is hanging straight down. Most of the work involved in writing the T and V is a matter of converting between cartesian and spherical coordinates. The formula for converting v^2 to spherical coordinates is:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta$$

Here, ϕ is the angle in the $x - y$ plane and θ is the angle from the positive z axis. Since r is constant ($r = l$) in the pendulum, this means that the kinetic energy is

$$T = \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\dot{\phi}^2 \sin^2 \theta)$$

The potential energy V should be zero when the pendulum is hanging down and maximum when it is standing up; in between, it moves with the cosine of θ:

$$V = mgl(1 + \cos \theta)$$

(If we had defined $\theta = 0$ as hanging down and $\theta = \pi$ as standing up, V would be $mgl(1 - \cos \theta)$ instead.) Combining these, we get:

$$\mathcal{L} = \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\dot{\phi}^2 \sin^2 \theta) - mgl(1 + \cos \theta)$$

The various derivatives are

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = ml^2\dot{\phi}^2 \sin \theta \cos \theta + mgl \sin \theta$$

[4] You can choose whatever variable name you want for the auxiliary variable; I randomly chose ζ here.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi}\sin^2\theta + 2ml^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta$$

Plugging those into Lagrange's equations yields

$$ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0$$

and

$$ml^2\ddot{\phi}\sin^2\theta + 2ml^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta = 0$$

These two second-order ODEs are the equations of motion for the system.

2.2 Symmetries and the connection to chaos

Note that the coordinate ϕ does not appear in the Lagrangian of the spherical pendulum. This is not just an algebraic technicality; it is a sign of a deep symmetry in the system. If a coordinate is absent from $\mathcal{L}$, it is called a *cyclic coordinate*. If you look at Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

you can see that ϕ's absence from $\mathcal{L}$ means that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

This means that there is a conserved quantity in the system — the *conjugate momentum*[5] $\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$. In this case, the conserved quantity is the angular momentum of the pendulum, and there is a corresponding symmetry in the system around the axis parametrized by ϕ.

In the pendulum, this symmetry is pretty obvious, but in other, more-complicated systems the existence of a cyclic coordinate may be the only way to tell if such a symmetry exists. In general, these conserved quantities are called *constants of the motion* or *integrals of the motion*; if $2n$ of them exist in an n degree-of-freedom system[6], then it is called *integrable* and cannot exhibit chaotic behavior. One of Michel Hénon's great contributions to dynamics was to prove that the three-body problem was nonintegrable.

2.3 Problems that involve other forces

If the system is affected by forces other than gravity (e.g., a mass on a spring, a charged pendulum moving in an environment where both magnetic and gravitational fields are active, or even a proton under the influence of the strong and weak forces), the potential V will have other terms besides the gravitational potential energy mgh. The potential energy stored in a spring, for example, is $\frac{1}{2}kx^2$, where x is how far the end of the spring is from its unloaded equilibrium position. For electromagnetic fields and relativistic velocities, potential energy is a lot more complicated; come see me if you're interested in deriving V for problems like this.

The point I want to emphasize here is that extremely hairy physics complications simply add terms to the V part (and, rarely, the T part) of the Lagrangian in a fairly straightforward way, and there's no need to change the seven-step procedure.

[5] "conjugate" to ϕ

[6] which has dimension $= 2n$

2.4 Hamiltonians

Deriving the equations of motion using the Hamiltonian (H) is a lot like the Lagrangian procedure, except that you use $\mathcal{H}(q_i, p_i) = T(q_i, p_i) + V(q_i, p_i)$ instead of $\mathcal{L}(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - V(q_i, \dot{q}_i)$ — where the *conjugate momentum* $p_i(t) = m\dot{q}_i(t)$ — and you plug and chug using Hamilton's equations:

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial \mathcal{H}}{\partial q_i} \end{aligned} \tag{2}$$

...instead of Lagrange's equations:

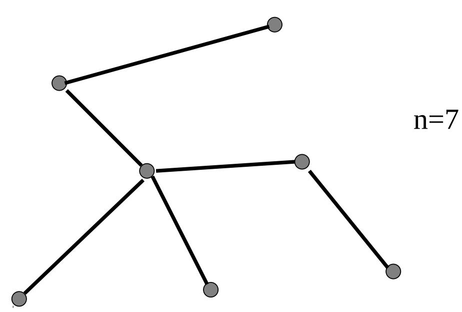
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

The results are also slightly different, but only in format; instead of n 2^{nd}-order ODEs, you get $2n$ 1^{st}-order ODEs.

3 Rigid-Body Dynamics

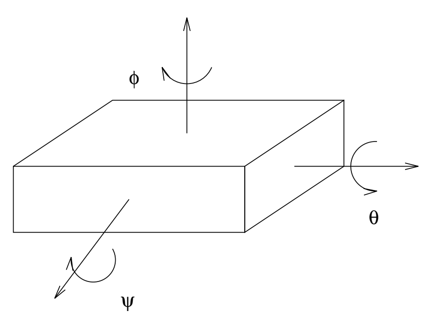
To describe the position of a single mass in three-space, you need three coordinates and six state-space axes (two for each coordinate). A system of n masses has $3n$ degrees of freedom (= coordinates) and thus has a $6n$-dimensional state space; its dynamics are governed by a set of $6n$ first-order ODEs.

If you put $n - 1$ rods between the masses in a manner that fixes them rigidly in space:



...the resulting *rigid body* only has six degrees of freedom: three for position and three for orientation. The rods act as *constraints* that reduce the dimension of the system, and the dynamics of the rotation/translation of the object are governed by a set of six second-order ODEs.

The standard way to specify the orientation of a rigid body is to use the Euler angles θ, ϕ, and ψ:



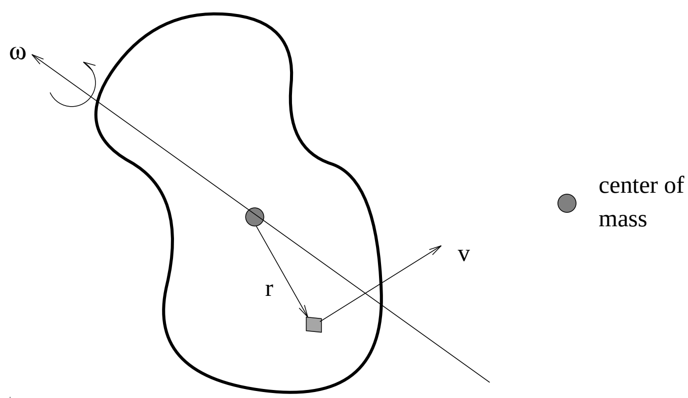
If you're sitting on the object itself — that is, you're in a *body-fixed* or *center-of-mass* frame of reference — θ, ϕ, and ψ are all you need to describe the system's state, and the equations of motion (EOMs) are a 6^{th}-order system instead of a 12^{th}-order one. If you're sitting in some *space-fixed* or *inertial* frame, you need to specify three position coordinates as well. We'll generally simplify things by assuming that friction plays no role and that nobody's applying any external torque to the object.

Given this framework, the way to think about the dynamics of rigid-body movement is to visualize giving an object some initial translational and angular velocities and watching how those velocities evolve. It's easier if you punt the translational velocity and concentrate on rotation. One way you can play with this is to put a rubber band around a book and throw it into the air, imparting some initial spin.

> *Experiment:* try spinning the book purely in the θ direction, then purely in the ϕ direction and purely in the ψ direction. Does it *stay* spinning in exactly the same way? You can also try starting it with some angular velocity that doesn't line up with these principal axes, but that makes things hard to watch.

The way to work out the mathematics that describes this process is to write down conservation of angular momentum, $\vec{L}$. Note that $\vec{L}$ is a *vector*, not a scalar; direction matters. An object of mass m located $\vec{r}$ away from some origin, O, moving with velocity $\vec{v}$, has angular momentum $\vec{L} = m\vec{r} \times \vec{v}$ around O. Angular momentum, like regular momentum, is a conserved quantity, as you'll explore in PS11. (That's how gyroscopes work.)

This kind of math requires some proficiency with vector calculus. For the purposes of refreshing your knowledge about this stuff, I'll start with a quick exercise. If you have an object rotating at some angular velocity $\vec{\omega}$ and you want to know how fast and in what direction some patch of stuff in that object is actually moving, you use the equation $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the vector from the center of mass to the patch of stuff. Recall that ω can have components in the θ, ϕ, and ψ directions. Here's a picture that should help you visualize what these vectors look like:



Note that the $\vec{\omega}$ vector must pass through the center of mass. Use the right-hand rule to visualize the cross-product: line your fingers up along $\vec{\omega}$ and rotate them until they point at $\vec{r}$. Your thumb now points at $\vec{v}$. To do this mathematically, use the 3×3 matrix equation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in the (x, y, z)-directions, respectively; these are also known as $\hat{x}$, $\hat{y}$, and $\hat{z}$.

I won't actually go through writing down conservation of angular momentum $\frac{d\vec{L}}{dt} = 0$; check out section 5.5–6 of Goldstein[3] if you're interested. Here are the results:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned} \quad (3)$$

These are called the {Euler-Lagrange-Poincaré-Hamill} equations, after the various people who had roles in their derivation. The I_i are the *principal moments of inertia*: how much the object "resists" rotation around each of its principal axes[7]. A cube, for instance, has $I_1 = I_2 = I_3$, a "triaxial" object like the one in the Euler angles picture has $I_1 > I_2 > I_3$, and an axisymmetric object like a top has $I_1 = I_2 \neq I_3$.

For a cube, the ELPH equations reduce to:

$$\begin{aligned} \dot{\omega}_1 &= 0 \\ \dot{\omega}_2 &= 0 \\ \dot{\omega}_3 &= 0 \end{aligned}$$

...which means that any initial angular velocity will remain unchanged forever. (Find a cubic object and try this out.)

For a triaxial object with, say, $I_1 = 1/2,\ I_2 = 1/3$ and $I_3 = 1/4$, the ELPH equations reduce to:

$$\begin{aligned} \dot{\omega}_1 &= \frac{1}{6}\omega_2\omega_3 \\ \dot{\omega}_2 &= -\frac{3}{4}\omega_3\omega_1 \\ \dot{\omega}_3 &= \frac{2}{3}\omega_1\omega_2 \end{aligned}$$

To do a traditional *stability analysis* of this system, I first find the fixed points, then linearize the equations around the fixed points and see what the eigenstuff says about how perturbations will shrink and/or grow. To find the fixed points, I set the equations to equal zero:

$$\begin{aligned} 0 &= \omega_2\omega_3 \\ 0 &= \omega_3\omega_1 \\ 0 &= \omega_1\omega_2 \end{aligned}$$

This condition is satisfied if any two of the ω_i are zero:

$$(\omega_1, \omega_2, \omega_3) = (0, 0, a),\ (0, a, 0),\ (a, 0, 0)$$

for any a. That is, the system has an infinite number of fixed points. If I linearize the ELPH equations around one of those points — say, the point $(1, 0, 0)$ — I get a set of three first-order linear ODEs that tell me about the local behavior near the fixed point:

$$\begin{aligned} \dot{\omega}_1 &= 0 \\ \dot{\omega}_2 &= -3/4\omega_3 \\ \dot{\omega}_3 &= 2/3\omega_2 \end{aligned}$$

If you look carefully at these equations, you'll see that they imply that ω_1 remains constant near this fixed point, and that the other two equations look a lot like a simple harmonic oscillator (that is, a perturbation in ω_2 will couple into ω_3 and vice versa, just like position and velocity in a mass on a spring). This kind of behavior identifies $(1, 0, 0)$ as an "elliptic fixed point." Small perturbations in ω_1 and/or ω_2 near such a point will not grow; rather, they will persist as a small, fixed-amplitude wobble. The $(0, 0, 1)$ point is also

[7] Principal axes are defined as those that diagonalize the moment-of-inertia tensor. If this makes no sense, don't worry. That's why it's in a footnote.

an elliptic fixed point, but if you linearize the ELPH equations around the fixed point $(0, 1, 0)$, you find that it's an *unstable* or *hyperbolic* point — like the inverted point in a pendulum. Perturbations around such a point will grow, so a small wobble will turn into a full-scale tumble very quickly. (Try this with your rubber-banded book.)

Axisymmetric objects, like tops or planets, have all sorts of interesting dynamics. If $I_1 = I_2 \neq I_3$, the ELPH equations become

$$\begin{aligned} \dot{\omega}_1 &= \Omega\omega_2 \\ \dot{\omega}_2 &= -\Omega\omega_1 \\ \dot{\omega}_3 &= 0 \end{aligned}$$

where $\Omega = \frac{I_1 - I_3}{I_1}$. The third equation says that the object's ω_3-type spin — along the spin axis of the top — will remain constant; the first and second say that the spin axis *precesses*. (See pp209–211 of Goldstein[3] for more information on this derivation.) Tops also *nutate* as they spin. Precession is the slow circular oscillation of the spin axis around the vertical; nutation is a "nodding" motion superimposed on that.

You can also use Lagrangians or Hamiltonians to derive the equations of motion for rotating axisymmetric objects. A top has three degrees of freedom, so we need three coordinates to describe it. The natural choice is to use the three Euler angles. The kinetic energy is

$$T = \frac{1}{2}I_1\dot{\phi}^2\sin^2\phi + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

(This calculation is **not** obvious! See any of the cited textbooks for a derivation.) The potential energy is $MgR\cos\theta$, where R is the radius from the point of the top to its center of mass. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}I_1\dot{\phi}^2\sin^2\phi + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

Note that ϕ and ψ are absent from this Lagrangian. Recall from section 2.2 that this means that these are "cyclic coordinates," and their existence implies that the system has two symmetries, along with two corresponding "constants of the motion." Recognizing this before diving into Lagrange's equations makes the algebra of the derivation of the equations of motion a bit easier.

The conditions under which the top precesses steadily — a situation that arises in physical tops and in the ELPH equations — can also be derived from these equations of motion; one looks for solutions in which the angle of inclination of the axis, θ, remains constant. I won't go through this, but it turns out that in this case both $\dot{\phi}$ and $\dot{\psi}$ must be constant, so the axis of the top precesses around the vertical with constant angular velocity Φ. For a given angle of inclination, there are two possible values for Φ.

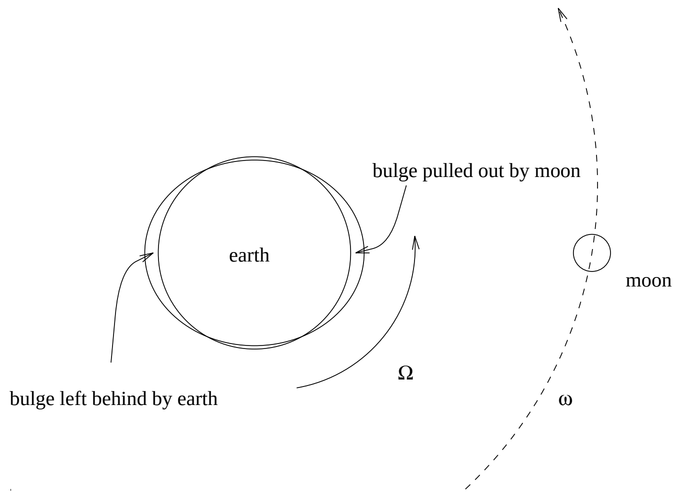
In the case of the earth, an oblate spheroid whose equatorial radius exceeds its polar radius by about 21.4 km — an axisymmetric top! — the angle of inclination is 23.5 degrees and $\Phi \approx \frac{2\pi}{26{,}000 years}$. This precession causes the pole of the Earth's axis to move relative to the fixed stars. To those of us who live on the surface, this is known as the "precession of the equinoxes" because the part of the sky that is directly behind the sun when it rises on the vernal equinox moves through the constellations of the Zodiac with a period of 26,000 years[8]. There are various interesting implications of the movement of the fixed stars around the sky in a circle; among other things, Polaris will not be the "north star" indefinitely, and one can date the pyramids by the small hole that was usually drilled out from the burial chamber so the pharoah could see Sirius during the afterlife. Earth also nutates, with a tenth of an arc-second amplitude and a period of 14 months. This is termed the "Chandler wobble." All of this (and much more) is covered in a wonderful book called *Newton's Clock: Chaos in the Solar System*[6], which is on reserve for CSCI 4446/5446.

Many interesting properties of the earth-moon system (or any other primary/satellite combination, for that matter) can be understood using the ideas in these notes. There are three stable equilibria to which

[8]Hence the "age of Aquarius" that began in the 1960s...

systems like this evolve — hence the "dark side of the moon" ,the fact that most of the planets in the solar system orbit around their longest axes, with those axes lined up perpendicular to the ecliptic.

The source of the Earth's tides, which occur twice a day, is the moon's pull: (1) on the ocean and (2) on the earth itself:



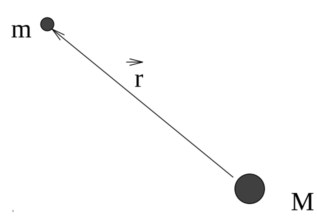
Though I didn't show these effects in the picture, the bulge on the moon side of the earth is bigger (which is why one of the two daily tides is higher than the other), and the earth pulls up tides in the moon's crust. The sun also plays a role; the *yearly* differences in the earth's tides — "spring" versus "neap" tides — are caused by the sun and moon cancelling out and then ganging up as they move around the earth.

Now, think about what happens if Ω is greater than ω: the bulges on the earth and mood will "get ahead" of the radial line joining the two objects, setting up a geometry where there is a differential pull that tries to get those bulges back in line with the earth-moon radius vector. The same thing happens if $\Omega < \omega$. The net result, over billions of years, has caused the system to evolve to a state where $\Omega = \omega$, so the same side of the moon always faces the earth. This happens in *any* primary/satellite pair and, in general, necessitates changing the energy of the bodies involved — through friction in their crusts and changes in orbital radii.

The mechanics of the second and third equilibria (spin axis lined up with the longest axis of the object and perpendicular to ecliptic) are beyond the scope of these notes; see Danby[2] for more details.

4 Gravitation

In classical mechanics, a point mass M pulls on another point mass m:



...in a manner described by the following equation:

$$\vec{F}_{mM} = \frac{-GmM\vec{r}}{|\vec{r}^3|}$$

(You probably learned this as $\frac{-GmM}{r^2}$; that form of the equation omits the direction vector.) "Equal and opposite" says that m pulls on M just as hard, and in the opposite direction:

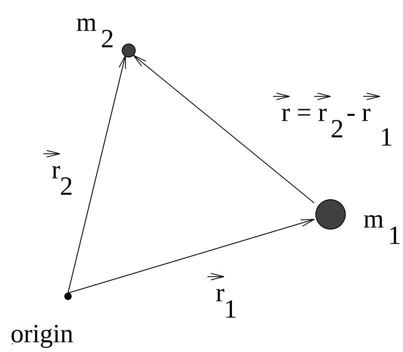
$$\vec{F}_{Mm} = \frac{GMm\vec{r}}{|\vec{r}^3|}$$

If really big masses, big velocities, or small length scales are involved, you have to use different equations — incorporating relativistic or quantum effects, respectively — but we won't worry about that here. Gravity is a *conservative* field. This means that a mass m accrues a certain amount of gravitational potential energy $mg\Delta h$ when you change its height by Δh, *no matter what path you take to do so.*

A good general reference for the stuff covered in this section is Danby[2], which is on library reserve for CSCI 4446/5446.

4.1 The Two-Body Problem

This section covers the derivation of the equations of motion for two bodies moving under the influence of their mutual attraction (i.e., no other forces like friction, electromagnetics, a nearby black hole, ...). Here's a picture that gives the setup and the terminology:



The force exerted on m_1 by m_2 is:

$$\vec{F}_{12} \quad = \quad -Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|(\vec{r}_1 - \vec{r}_2)|^3}$$

Since $\vec{F}_{12} = m_1\vec{a}_1$ and $\vec{a}_1 = m_1\ddot{\vec{r}}_1$, the equation of motion of m_1 is

$$m_1\ddot{\vec{r}}_1 \quad = \quad -Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|(\vec{r}_1 - \vec{r}_2)|^3}$$

You get the other equation of motion by flipping the indices:

$$m_2\ddot{\vec{r}}_2 \quad = \quad -Gm_2m_1\frac{\vec{r}_2 - \vec{r}_1}{|(\vec{r}_2 - \vec{r}_1)|^3}$$

Pulling these apart into a system of first-order ODEs yields:

$$\begin{aligned} \dot{\vec{r}}_1 &= \vec{v}_1 \\ \dot{\vec{v}}_1 &= -Gm_2\frac{\vec{r}_1 - \vec{r}_2}{|(\vec{r}_1 - \vec{r}_2)|^3} \\ \dot{\vec{r}}_2 &= \vec{v}_2 \\ \dot{\vec{v}}_2 &= -Gm_1\frac{\vec{r}_2 - \vec{r}_1}{|(\vec{r}_2 - \vec{r}_1)|^3} \end{aligned} \tag{4}$$

These equations may look like a fourth-order system, but recall that $\vec{r}$ and $\vec{v}$ are three-vectors — $\vec{r} = (x\hat{x} + y\hat{y} + z\hat{z})$ and $\vec{v} = (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$ — so equations (4) really describe a 12^{th}-order system. The second equation in the system above, for instance, is really

$$(\ddot{x_1}\hat{x} + \ddot{y_1}\hat{y} + \ddot{z_1}\hat{z}) = -Gm_2 \frac{(x_1\hat{x} + y_1\hat{y} + z_1\hat{z}) - (x_2\hat{x} + y_2\hat{y} + z_2\hat{z})}{[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{3/2}}$$

To pull this equation apart into its three constituent equations, you take all the $\hat{x}$ parts from the left-hand side and equate them to the $\hat{x}$ parts from the right-hand side:

$$\ddot{x}_1 = -Gm_2 \frac{(x_1 - x_2)}{[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{3/2}}$$

...then repeat for $\hat{y}$ and $\hat{z}$ to get the other two equations. The first and third equations in the system (4) are much simpler, but you'll need to define some symbols for the components of $\vec{v}$:

$$\begin{aligned}
\dot{\vec{r}}_1 &= \vec{v}_1 \\
(\dot{x}_1\hat{x} + \dot{y}_1\hat{y} + \dot{z}_1\hat{z}) &= (u_1\hat{x} + v_1\hat{y} + w_1\hat{z}) \\
\dot{x}_1 &= u_1 \\
\dot{y}_1 &= v_1 \\
\dot{z}_1 &= w_1
\end{aligned}$$

The standard (and useful) way to simplify the two-body equations is to use *relative* coordinates: pretend you're sitting on one of the masses and write an equation to describe what you see the other one doing. If I stick the origin of my coordinate system at m_1, add equations (4) together, and use the vector $\vec{r} = \vec{r}_2 - \vec{r}_1$ to rewrite the result, I get:

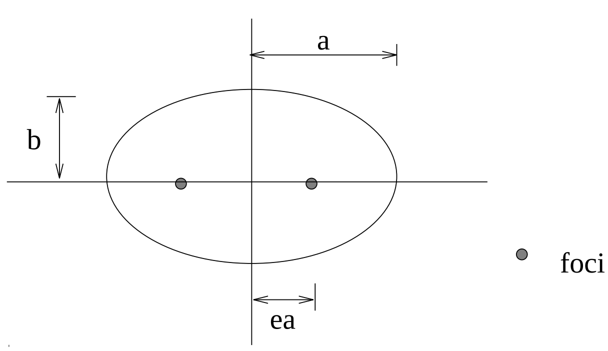
$$\begin{aligned}
\dot{\vec{r}} &= \vec{v} \\
\dot{\vec{v}} &= -\gamma \frac{\vec{r}}{|\vec{r}|^3}
\end{aligned} \qquad (5)$$

...where $\gamma = G(m_1 + m_2)$.

Note that the ODE systems (4) and (5) have similar forms, so their solutions will be similar. These solutions are fairly easy to figure out and not at all chaotic; they take the form of *conic sections*: ellipses, parabolae, and hyperbolae. The implications of the close resemblance between the systems (4) and (5) are interesting and somewhat hard to visualize: each of the two bodies is travelling on a conic section around the other one. Though I haven't shown the equations that justify it, it is also the case that *both* bodies are on conic-section orbits around the center of mass.

One of Kepler's Big Contributions — his first law — was a special case of this revelation about conic sections being the only possible solution to the two-body problem: he figured out that planets follow elliptical paths around the sun. The story of this is quite interesting; check out chapter 3 of [6] or the middle third of *The Sleepwalkers*[5] if you're interested.

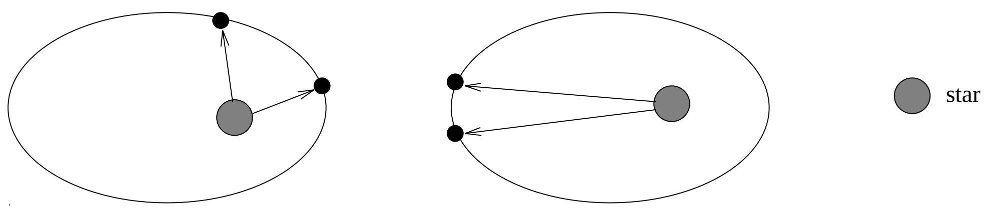
Most of the orbits we'll play with are ellipses. Here's a picture that defines the geometry of an ellipse:



a and b are called the *semimajor* and *semiminor* axes, respectively. The higher the eccentricity, e, the skinnier the ellipse. The eccentricities of the orbits of the planets in the solar system are tabled below for your interest:

mercury	0.206	saturn	0.056
venus	0.007	uranus	0.046
earth	0.017	neptune	0.010
mars	0.093	pluto	0.248
jupiter	0.048		

Kepler's second law says that a body on an elliptic orbit "sweeps out equal areas in equal times." That is, if you take pictures of some planet • at two different times Δt apart:



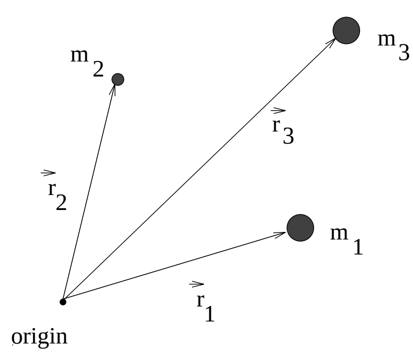
...the areas between the ellipse and the two radii are equal. Among other things, this means that an orbiting body moves faster near *periapse*, the point of closest approach, and slowly when it's far away from its primary. Astrophysicists typically specify where something is on its orbit using the *orbital phase*. This is typically quantified by *true anomaly*: the angle of the current position measured from periapse.

Kepler's *third* law was the relationship between period and semimajor axis: $P^2 = a^3$. Note that this only works if you use the right units: years for the former and AUs (astronomical units; the mean distance between the sun and the earth. $1AU \approx 149,600,000km$).

4.2 The Three-Body Problem

The three-body equations are just like the two-body equations except half again as complicated[9]. In behavior, they're *completely* different; the latter only have conic-section solutions, whereas the former can exhibit chaotic behavior.

Here's a picture of the setup and notation:



The equations are:

$$\ddot{\vec{r}}_1 = -G\left(m_2\frac{\vec{r}_1-\vec{r}_2}{|(\vec{r}_1-\vec{r}_2)|^3}+m_3\frac{\vec{r}_1-\vec{r}_3}{|(\vec{r}_1-\vec{r}_3)|^3}\right) \quad (6)$$

[9] 18^{th}-order instead of 12^{th}-order

$$\ddot{\vec{r}}_2 = -G\left(m_1\frac{\vec{r}_2-\vec{r}_1}{|(\vec{r}_2-\vec{r}_1)|^3}+m_3\frac{\vec{r}_2-\vec{r}_3}{|(\vec{r}_2-\vec{r}_3)|^3}\right)$$
$$\ddot{\vec{r}}_3 = -G\left(m_1\frac{\vec{r}_3-\vec{r}_1}{|(\vec{r}_3-\vec{r}_1)|^3}+m_2\frac{\vec{r}_3-\vec{r}_2}{|(\vec{r}_3-\vec{r}_2)|^3}\right)$$

PS13 covers writing out these equations and solving them numerically. The article by Hut and Bahcall[4] (which in the 1999 course pack) shows some of the interesting behavior that can arise in three-body interactions.

References

[1] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 1989. Second edition.

[2] J. M. A. Danby. *Fundamentals of Celestial Mechanics*. William Bell, Richmond VA, 1962. 3rd edition, 1992.

[3] H. Goldstein. *Classical Mechanics*. Addison Wesley, Reading MA, 1980.

[4] P. Hut and J. N. Bahcall. Binary-single star scattering: I. numerical experiments for equal masses. *Astrophysical Journal*, 268:319–341, 1983.

[5] A. Koestler. *The Sleepwalkers*. Macmillan, New York, 1959.

[6] I. Peterson. *Newton's Clock: Chaos in the Solar System*. W. H. Freeman, New York, 1993.