Nonlinearity,
11:913-922
(1998)

Computing Connectedness: an exercise in computational topology

V. Robins, J.D. Meiss and E. Bradley
Department of Applied Mathematics
University of Colorado
Boulder, CO 80309

June 15, 2000

Abstract

We reformulate the notion of connectedness for compact metric spaces in a manner that may be implemented computationally. In particular, our techniques can distinguish between sets that are connected; have a finite number of connected components; have infinitely many connected components; or are totally disconnected. We hope that this approach will prove useful for studying structures in the phase space of dynamical systems.

AMS classification scheme numbers: 54D05, 58F13, 68U05.

Published in: *Nonlinearity* 11:913–922, 1998.

1 Introduction

The extraction of qualitative features from data is one of the primary goals of experimental science. For data that have a natural geometrical representation, or whose geometry arises from phase space embedding [1, 2, 3], such properties can be obtained by either metrical or topological techniques. Metric structure is often probed by the computation of fractal dimensions [4] or Lyapunov exponents [5]. Other approaches include the estimation of Hausdorff dimension using minimal spanning trees [6] and the concept of local connected fractal dimension [7]. Topological properties, though more fundamental than metrical ones, are more difficult to extract from data.

Examples of topological techniques include the calculation of symbolic dynamics of a flow (for the case when there is an attractor that can be embedded in $\mathbb{R}^3$) by analysis of its "template" [8, 9]; and the computation of homology from flows that lie on smooth manifolds [10, 11, 12].

By "computational topology" we mean the study of topological properties of an object that can be computed to some finite accuracy. Often we can improve the accuracy of the computations at the expense of increased computer or experimental effort, and the algorithms we study lead to topology by extrapolation.

This notion can be distinguished from "digital topology" [13], which deals with the topological features of digital images, given by discrete values (typically binary) on a regular grid (typically two-dimensional and square). This important field has many applications, including algorithmic pattern recognition, which plays an important role in computer vision (e.g., determining whether a robot-width corridor exists between two obstacles [14]) and remote sensing (e.g., computing the boundaries of a drainage basin from satellite data [15]). The fundamental concept in this field is that of adjacency, the definition of which depends upon the lattice. Much work in this area has focussed on algorithms for the labelling of components [13], boundaries [16], and other features of digital images. Basic results include consistent notions for connectedness [13], simple connectedness [17], a digital Jordan curve theorem [18], and algorithms for the Euler characteristic [19, 20] of digital sets.

In this paper we consider notions of connectedness that can be computed by extrapolation from computations with finite precision. We show that Cantor's definition of connectedness leads to an algorithm that is guaranteed to decide if a compact set is connected providing the computations can be continued to arbitrary accuracy. More practically, even when a set can only be represented with finite precision, we can use extrapolation to distinguish between sets that appear to be connected, have a finite number of components, have an infinite number of components, or be totally disconnected.

We propose that sets that have an infinite number of components can be characterized by a rate of growth of the number of components with improving resolution, we call this the disconnectedness index, γ. This rate is analogous to the box-counting dimension; for the simplest such sets, e.g. the middle-thirds Cantor set, γ is numerically equal to the dimension. However, more generally the disconnectedness index distinguishes between sets with the same dimension.

Similarly, sets with components of zero diameter can be characterized by a discreteness index, δ, the rate of decrease of the diameter of the components with resolution.

We give a number of simple examples in this paper. In a paper to follow, we will apply these methods to numerical data. A prime motivation for these algorithms is to study the destruction of invariant tori in maps of the torus or in symplectic maps. It is well known that for circle homeomorphisms or area-preserving twist maps, when an invariant circle is destroyed it becomes an invariant cantor set. The topology of the analogous objects in higher dimensions is not known. Numerical studies may help lead to an understanding of these issues.

2 Connectedness

Connectedness is a very intuitive concept: is the space one contiguous piece or not? The generally accepted definition is that a topological space X is *connected* if and only if it cannot be decomposed into the union of two non-empty, disjoint, closed sets. If such a decomposition exists then X is said to be *disconnected* — that is, if there are two closed sets U and V such that $U \cap V = \emptyset$ and $U \cup V = X$.

Our aim is to reformulate the notion of connectedness in a way that relies on extrapolation, in order to make it possible to implement a test for this property on a computer. The basic idea is to look at the set with a finite resolution ϵ and see how certain properties change as we let $\epsilon \to 0$.

Given a subset X, of a metric space, we say it is *ϵ-disconnected* if it can be decomposed into two sets that are separated by a distance of at least ϵ — i.e. there are two closed subsets,

U and V with $U \cup V = X$ and $d(U,V) = \inf_{x \in U, y \in V} d(x,y) > \epsilon$. Otherwise, X is *ϵ-connected.* One would now hope that if X is ϵ-connected for all $\epsilon > 0$, then X is connected in the standard sense. This is in fact true, provided that X is also compact, which is easily demonstrated by proving the contrapositive:

Lemma 1. *If a compact metric space X is disconnected, it is ϵ-disconnected for some $\epsilon > 0$.*

Our definition is equivalent to an early formulation of connectedness in metric spaces due to Cantor [21] that uses the concept of an *ϵ-chain*: a finite sequence of points $x_0, \ldots, x_n$ that satisfy $d(x_i, x_{i+1}) < \epsilon$ for $i = 1, \ldots, n$. Cantor calls a set X connected when every two points in X can be linked by an ϵ-chain for arbitrarily small ϵ. We refer to this property as *Cantor-connected.*

An example which illustrates the necessity of compactness is the set of rational numbers. They are Cantor-connected but are disconnected in the regular sense. The restriction to compact metric spaces is not as bad as it might seem, since we are primarily interested in sets that are generated dynamically, such as ω-limit sets of orbits, and these sets are compact when they are bounded.

The objects we are most interested in are the connected components of a set. We say S is an *ϵ-component* of X if $S \subset X$ is ϵ-connected and $d(X \setminus S, S) > \epsilon$. Given a resolution, ϵ, X has a natural decomposition as the disjoint union of its ϵ-components. The central idea behind this paper is that investigating how this decomposition changes as ϵ approaches zero tells us about the connectedness of X. For example, if the only ϵ-component is X itself for all ϵ, then we can use lemma 1 to conclude that X is connected. The theorem below tells us what happens in the more-complex case that X is *totally disconnected* — i.e. if the connected component of every point $x \in X$ is exactly $\{x\}$. Recall that the diameter of a set is defined as $\mathrm{diam}(S) = \sup_{x,y \in S} d(x,y)$. We have then:

Theorem 2. *Let X be a compact set and let S_ϵ represent an ϵ-component of X. Then X is totally disconnected if and only if*

$$\lim_{\epsilon \to 0} \sup_{S_\epsilon} \mathrm{diam}(S_\epsilon) = 0 \ .$$

The proof of this theorem is straightforward and uses the following lemma, which is a direct consequence of compactness.

Lemma 3. *If X is a compact set, it has a finite number of ϵ-components.*

Proof of the Theorem. By way of obtaining a contradiction to the forward direction, suppose that

$$\lim_{\epsilon \to 0} \sup_{S_\epsilon} \mathrm{diam}(S_\epsilon) = \delta > 0.$$

Take any sequence of $\epsilon_n \to 0$ and construct a tree as follows. At level n list all the ϵ_n-components with diameter $\geq \delta$. There are a finite number of these by the lemma. Order the tree by set inclusion, i.e. an edge connects S_{ϵ_j} and $S_{\epsilon_{j+1}}$ if and only if $S_{\epsilon_{j+1}} \subset S_{\epsilon_j}$. We know that there must be ϵ-components with diameters $\geq \delta$ for all ϵ, so this tree must have an infinite branch. This gives a sequence of *nested* components S_{ϵ_n}, with $\mathrm{diam}(S_{\epsilon_n}) \searrow \delta$. Since the sets are nested, they have a limit, $S_* = \lim_{n \to \infty} S_{\epsilon_n} = \bigcap_n S_{\epsilon_n}$. It then follows that S_* is

Table 1: Summary of the connectedness properties of a set that can be deduced from the limiting behavior of $C(\epsilon)$ and $D(\epsilon)$.

	$C(\epsilon) \equiv 1$	$\lim_{\epsilon \to 0} C(\epsilon) < \infty$	$\lim_{\epsilon \to 0} C(\epsilon) = \infty$
$\lim_{\epsilon \to 0} \max D(\epsilon) = 0$	A single point	A finite, totally disconnected set	An infinite, totally disconnected set
$\lim_{\epsilon \to 0} \max D(\epsilon) > 0$	A connected set, $D(\epsilon) \equiv \mathrm{diam}(X)$	A finite number of connected components	Infinitely many connected components

ϵ-connected for all ϵ and $\mathrm{diam}(S_*) = \delta$. This implies S_* has at least two points in it and so X could not be totally disconnected.

The converse follows essentially from the definition. If all diameters of the ϵ-components go to zero then the connected component of any $x \in X$ must be $\{x\}$ so X is totally disconnected. □

Theorem 2 and lemma 3 motivate the introduction of two important quantities that are used to classify the connectedness properties of a set: (1) the number of ϵ-components, $C(\epsilon)$, and (2) the set of diameters of the ϵ-components, $D(\epsilon)$. The behavior of these quantities as $\epsilon \to 0$ tells us whether or not the set is totally disconnected and how many connected components there are. The different cases are summarized in Table 1. In the next section, we present simple examples that illustrate some of the possible types of set and demonstrate the application of our results. Understanding the behavior of $C(\epsilon)$ and $D(\epsilon)$ for these isotropic sets provides a foundation for investigating more-general sets that are likely to be a union of these basic types.

3 Examples

We focus on two groups of examples: one-dimensional Cantor sets in section 3.1 and relatives of the Sierpinski gasket in 3.2. Both were chosen because they are easily described in terms of ϵ-components and are therefore good test cases to illustrate the approaches described in the previous section. Cantor sets are important structures in dynamics — in particular, they arise in circle homeomorphisms and area-preserving twist maps. The Sierpinski relatives are two-dimensional attractors of a family of iterated function systems and are described at length in Peitgen *et al.* [22]. They include a range of topological structures that we might expect to see in invariant sets of higher-dimensional dynamical systems. In section 3.3 we extract principles from these examples that hold for general iterated function systems.

3.1 Cantor Sets

A Cantor set is totally disconnected, perfect and compact. Recall that a set is *perfect* if it is equal to the set of its accumulation points. In other words, every point has arbitrarily small neighbourhoods containing infinitely many other points, so no point is isolated. This can be formulated in our ϵ-resolution terms as:

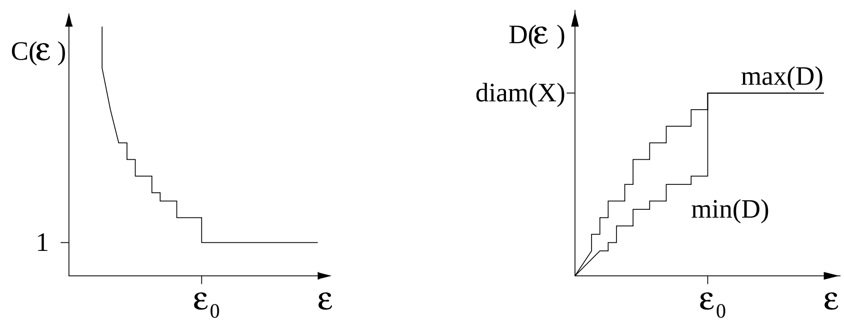

Figure 1: The general behavior of $C(\epsilon)$ and $D(\epsilon)$ for a Cantor set.

Lemma 4. *A compact set X is perfect if and only if the* $\inf_{S_\epsilon} \operatorname{diam}(S_\epsilon) > 0$ *for all* $\epsilon > 0$.

This should be clear from the definitions.

To simplify notation a little, let $\max D(\epsilon) = \overline{D(\epsilon)}$ and $\min D(\epsilon) = \underline{D(\epsilon)}$.

In fig. 1 we show schematically how we expect $C(\epsilon)$ and $D(\epsilon)$ to behave as $\epsilon \to 0$. Theorem 2 tells us that $\overline{D(\epsilon)} \to 0$; since a Cantor set is both perfect and totally disconnected, we must have $C(\epsilon) \to \infty$. A more-accurate description is given by the asymptotic behavior, which we assume to be a general power law. That is, near $\epsilon = 0$, $\overline{D(\epsilon)} \sim \epsilon^\delta$ and $C(\epsilon) \sim \epsilon^{-\gamma}$. The exponents may be found as the following limits:

$$\begin{aligned} \delta &= \liminf_{\epsilon \to 0} \frac{\log \overline{D(\epsilon)}}{\log \epsilon} \\ \gamma &= \liminf_{\epsilon \to 0} \frac{\log C(\epsilon)}{\log(1/\epsilon)} \end{aligned}$$

We call δ the *discreteness* index. The limit is taken as the lim inf so that we know $\overline{D(\epsilon)} \le D_0 \epsilon^\delta$ for some constant D_0. We think of δ as measuring how sparsely the points are distributed — in fact, it is loosely related to the notion of thickness for one-dimensional Cantor sets. The component growth rate, γ, is called the *disconnectedness* index. It is also found as the $\liminf$ to ensure $C(\epsilon) \ge C_0 \epsilon^{-\gamma}$ for some constant C_0. In practice, these indices may be computed using a particular sequence of ϵ-values, in an analogous manner to calculations of the box-counting dimension [23].

Note that these indices are invariant under bi-Lipshitz homeomorphisms. They are not true topological invariants because they are defined in terms of metric quantities.

Middle-α Cantor sets

These Cantor sets have zero Lebesgue measure and arise in piecewise-linear, one-dimensional maps. Let $0 < \alpha < 1$ and consider the Cantor set $K_\alpha \subset [0,1]$ constructed by successively removing the middle α-proportion of each remaining interval. This construction has a natural correspondence with the ϵ-components. At a given level n, there are $C_n = 2^n$ intervals of equal length $D_n = \frac{1}{2}(1-\alpha)D_{n-1}$, separated by gaps of at least $g_n = \alpha D_{n-1}$; see fig. 2. With $D_0 = 1$, the recursion relations may be solved to find $D_n = [\frac{1}{2}(1-\alpha)]^n$ and $g_n = \alpha[\frac{1}{2}(1-\alpha)]^{n-1}$, so

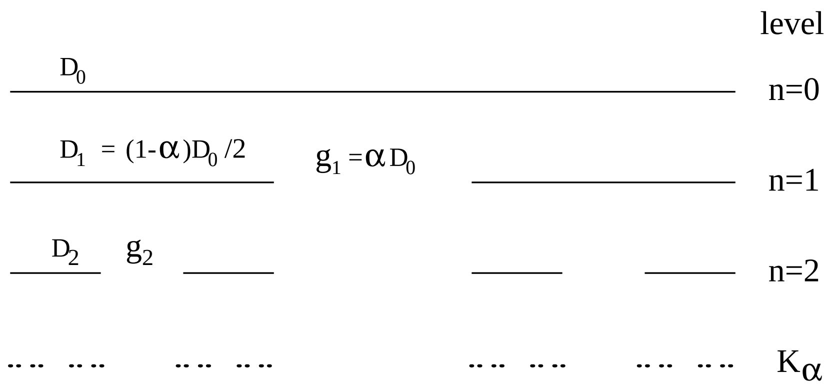

Figure 2: The construction of a K_α Cantor set.

that

$$\delta = \lim_{n\to\infty} \frac{\log D_n}{\log g_n} = \lim_{n\to\infty} \frac{n \log[\frac{1}{2}(1-\alpha)]}{(n-1)\log[\frac{1}{2}(1-\alpha)] + \log\alpha} = 1 \ ,$$

and

$$\gamma = \lim_{n\to\infty} -\frac{\log C_n}{\log g_n} = \lim_{n\to\infty} \frac{-n\log 2}{(n-1)\log[\frac{1}{2}(1-\alpha)] + \log\alpha} = \frac{\log 2}{\log 2 - \log(1-\alpha)} \ .$$

The discreteness index, δ, is independent of α because the Cantor set is constructed in such a way that the ϵ-components and gaps decrease at the same rate. The disconnectedness index, γ, has the same value as the Hausdorff dimension.

A Cantor set with positive measure

Now consider a Cantor set with gaps that decrease more rapidly. Let K be the subset of $[0,1]$ obtained by successively removing gaps from the center of remaining intervals, with widths $g_n = (\frac{1}{2})^{2n-1}(\frac{1}{10})$. The total Lebesgue measure of the gaps is just $\frac{1}{10}$, so the measure of K is $\frac{9}{10}$. After a bit of algebra, we find that for $(\frac{1}{2})^{2n+1}(\frac{1}{10}) \le \epsilon < (\frac{1}{2})^{2n-1}(\frac{1}{10})$, there are $C_n = 2^n$ components with diameters $D_n = (\frac{1}{2})^n[\frac{9}{10} + (\frac{1}{2})^n(\frac{1}{10})]$. So,

$$\delta = \lim_{n\to\infty} \frac{n\log(\frac{1}{2}) + \log[\frac{9}{10} + (\frac{1}{2})^n(\frac{1}{10})]}{(2n-1)\log(\frac{1}{2}) + \log(\frac{1}{10})} = \frac{1}{2} \ ,$$

and

$$\gamma = \lim_{n\to\infty} \frac{-n\log 2}{(2n-1)\log(\frac{1}{2}) + \log(\frac{1}{10})} = \frac{1}{2} \ .$$

Since K has positive Lebesgue measure, its Hausdorff dimension is 1. Thus, as noted earlier, the disconnectedness index and the dimension are distinct.

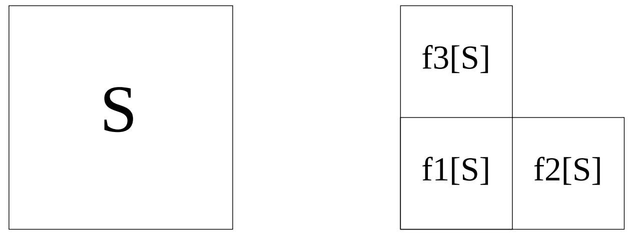

Figure 3: The general template for a Sierpinski relative.

3.2 Relatives of the Sierpinski gasket

The Sierpinski gasket, S, is a fractal that is the attractor of the simple iterated function system (IFS): $S = f[S] = f_1[S] \cup f_2[S] \cup f_3[S]$, where the f_i are similarity transformations with contraction ratio $\frac{1}{2}$:

$$
\begin{aligned}
f_1(x,y) &= \tfrac{1}{2}(x,y)\\
f_2(x,y) &= \tfrac{1}{2}(x+1,y)\\
f_3(x,y) &= \tfrac{1}{2}(x,y+1).
\end{aligned}
$$

The "relatives" of S are generated by composing each of the f_i with one of the eight symmetries of the square. The general template is shown in fig. 3. This gives, in principle, $8^3 = 512$ different fractals; due to symmetry considerations, however, there are in fact only 232. They all have the same Hausdorff dimension of $\log 3/\log 2$, and yet there is a great deal of variation in their topological structure. Peitgen *et al.* [22] identify three different classes: one-dimensional and connected like the gasket; simply connected (i.e. homotopic to a point), or totally disconnected and therefore zero dimensional. There is, in fact, a fourth type that has infinitely many connected components, but a topological dimension of 1. Examples of each type are shown in fig. 4. Our classification scheme cannot distinguish between the different types of connected set: for both the connected cases, we have $C(\epsilon) \equiv 1$ and $D(\epsilon) \equiv 1$. The next two examples are of the disconnected types.

A 2-D Cantor Set

In fig. 4(c) we show the fractal generated by:

$$
\begin{aligned}
f_1(x,y) &= \tfrac{1}{2}(-y+1,x)\\
f_2(x,y) &= \tfrac{1}{2}(y+1,x)\\
f_3(x,y) &= \tfrac{1}{2}(y,-x+2)
\end{aligned}
$$

The structure of this set can be analyzed by finding its ϵ-components. We call the region that disconnects two ϵ-components a "gap". Its "width" is the metric distance between the two components. Clearly, the set has a single component whenever ϵ is greater than the width of the largest gap, g_0. Using the L_∞ metric, we find $g_0 = \frac{1}{4}$. As ϵ decreases, subsequent components are resolved at gap sizes $g_n = g_0/2^n$. The diameters of the components follow the same pattern after some initial transient behavior. Measured in the L_∞ metric: $\overline{D_0} = 1, \overline{D_1} = \overline{D_2} = \frac{1}{2}, \overline{D_3} = \frac{1}{4}, \ldots, \overline{D_n} = (\frac{1}{2})^{n-1}$. This gives $\delta = 1$, showing that the set is totally disconnected.

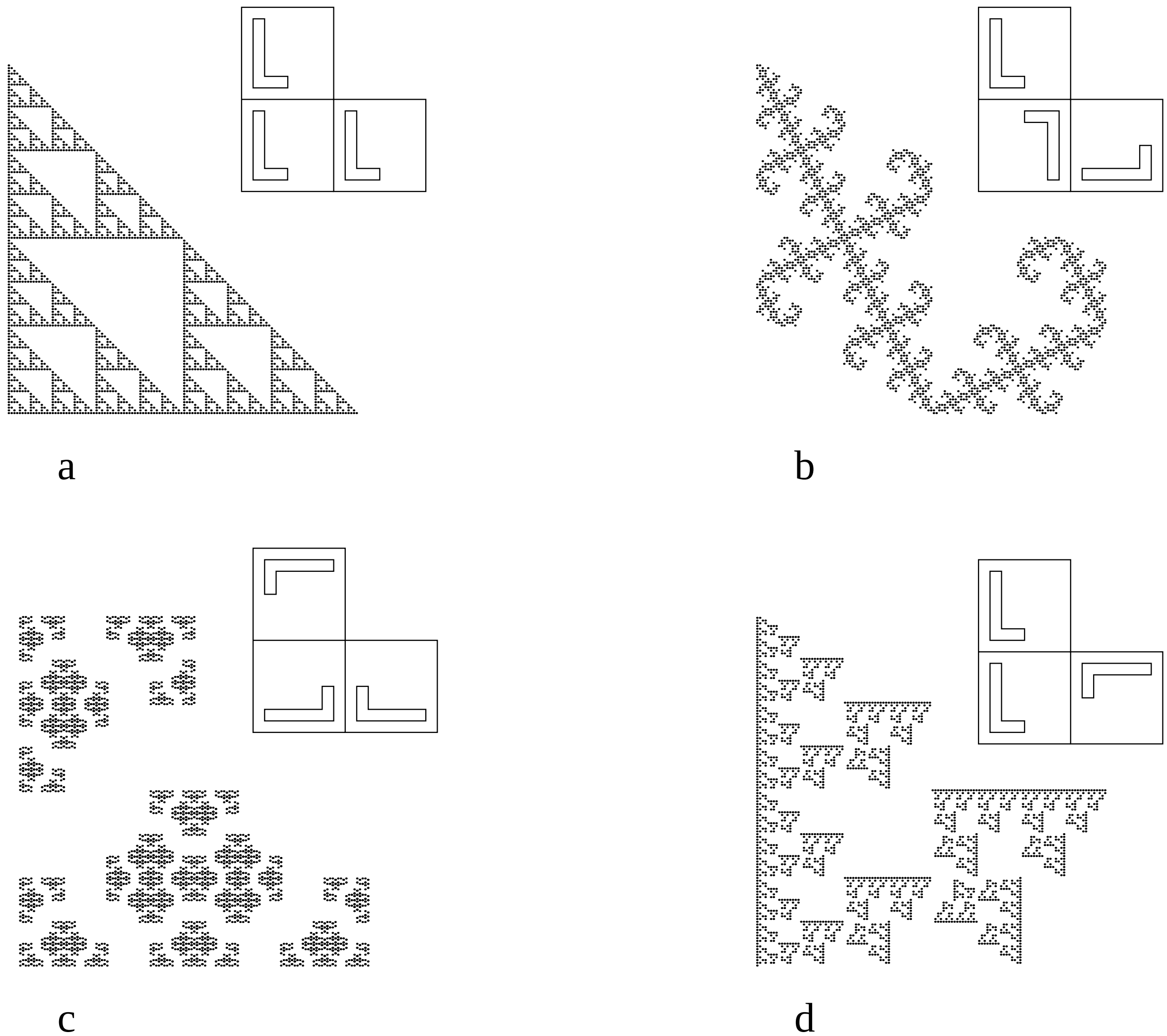

Figure 4: Four Sierpinski relatives and their templates (the L shows the orientation of the image of S under each similarity). (a) The Sierpinski gasket; (b) A gasket relative that is simply connected; (c) One that is totally disconnected; (d) A relative with infinitely many connected components.

The number of components, C_n, is a little harder to determine. Since C_n is just one more than the total number of gaps, we calculate it by first deriving an expression for the latter. Let N_n be the number of gaps of size g_n. Since the fractal contains three copies of itself, one might think that $N_n = 3^n$. By careful inspection of the template we find that this is not the whole story — some gaps merge into one. In fact, with $N_0 = 1$ we have the recursion:

$$N_n = \begin{cases} 3N_{n-1} & \text{if } n \text{ is odd,} \\ 3N_{n-1} - 2 \cdot 3^{n/2-1} & \text{if } n \text{ is even.} \end{cases}$$

These are solved to find:

$$N_n = \begin{cases} 2 \cdot 3^{n-1} + 3^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 2 \cdot 3^{n-1} + 3^{n/2-1} & \text{if } n \text{ is even.} \end{cases}$$

We then compute

$$C_n = 1 + \sum_{j=0}^{n} N_j = \begin{cases} 3^n + 2 \cdot 3^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 3^n + 3^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

The leading power is the same for all n, so we may use either case to evaluate the limit:

$$\gamma = \liminf_{n\to\infty} \frac{\log(C_n)}{\log(1/g_n)} = \lim_{n\to\infty} \frac{\log[3^n + 3^{n/2}]}{\log[2^n/g_0]} = \frac{\log 3}{\log 2} \,.$$

Since there are no isolated points, the attractor is perfect and therefore a Cantor set as claimed.

A set with infinitely many connected components

The gasket relative in fig. 4(d) is a fractal that illustrates the case from Table 1 wherein S has infinitely many connected components and yet is not totally disconnected. It has the same component growth rate as the Cantor set above. The similarities for the IFS are:

$$\begin{aligned} f_1(x,y) &= \tfrac{1}{2}(x,y) \\ f_2(x,y) &= \tfrac{1}{2}(y+1, -x+1) \\ f_3(x,y) &= \tfrac{1}{2}(x, y+1) \end{aligned}$$

Again the gaps decrease simply as $g_n = g_0/2^n$. This time, the number of components is just $C_n = 1 + \sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1}+1)$, giving $\gamma = \log 3/\log 2$. The ϵ-component diameters, however, do not decrease with the resolution. In fact, using the L_∞ metric, the set of diameter values is $D_n = \{1, \frac{1}{2}, \dots, (\frac{1}{2})^n\}$; so $\overline{D_n} = 1$ for all n.

Thus, our ϵ-component representation has made clear the topological distinction between these two disconnected sets.

3.3 General Iterated Function Systems

Iterated function systems have the general form: $S = f[S] = f_1[S] \cup f_2[S] \cup \ldots \cup f_m[S]$ where each f_i is an affine transformation. In the case that the f_i are similarities, each has a uniform contraction ratio $0 < r_i < 1$. Since $S = f[S]$, if two connected components of S are separated by a gap of size g_0, then there must be infinitely many separating gaps with sizes $r_1^{s_1} r_2^{s_2} \cdots r_m^{s_m} g_0$ for all integers $s_i \geq 0$. Similarly, if there is an ϵ-connected component of diameter D_0, then there are components (at other resolutions) with diameters $r_1^{s_1} r_2^{s_2} \cdots r_m^{s_m} D_0$. In the case that all the similarity ratios are the same, as with the middle-α Cantor sets and Sierpinski relatives, this simplifies dramatically to $g_n = r^n g_0$ and $D_n = r^n D_0$, where n is a (sufficiently large) integer. Any such IFS whose attractor is a Cantor set must therefore have $\delta = 1$. This is exactly what was found in the examples above. The measure-$\frac{9}{10}$ Cantor set has $\delta = \frac{1}{2}$ because the gap sizes decrease more rapidly than the diameters. This means it cannot be described as the attractor of an IFS consisting of similarities only (although it may be possible to do this if more general functions are allowed).

4 Conclusions

We have reformulated the definition of connectedness for the case of compact metric spaces. By looking at a set with increasingly fine resolution we can determine whether it is connected; has a finite or infinite number of connected components; or is totally disconnected. This is accomplished by determining the limiting behavior of two readily computable properties — the number of ϵ-components, $C(\epsilon)$, and their diameters, $D(\epsilon)$. Details of the computer implementation of these results will be the topic of a future paper.

Other topological properties that would be nice to compute include the notion of "perfect." We have already rephrased the definition in ϵ-component terms. This could be refined further, to some notion of a rate analogous to the discreteness index. It would also be useful to distinguish between the different "flavors" of connectedness — particularly to tell if the set contains holes.

It should also be straight-forward to adapt our techniques to compute the "thickness" of Cantor sets. This figures prominently in Newhouse's theory of the persistence of homoclinic tangencies [24].

Our approach to computing such topological properties is more flexible than that of digital topology. Objects are not confined to a regular grid and may be represented by a general point-set of arbitrary dimension. This framework will enable us to investigate the structure of attractors and other important sets in dynamical systems. It will be particularly useful when the phase space has dimension greater than 3 as it becomes very difficult to visualize such objects.

Aknowledgements

Useful conversations with R. Easton are gratefully acknowledged. VR thanks the School of Mathematical Sciences at the Australian National University for use of their facilities while writing the paper. JDM was supported in part by NSF grant number DMS-9623216. EB and VR were supported by NSF National Young Investigator award #CCR-9357740; EB was also supported by ONR #N00014-96-1-0720 and a Packard Fellowship in Science and Engineering.

References

[1] N.H. Packard, J.P. Crutchfield, J.D. Farmer, and R.S. Shaw. Geometry from a time series. *Phys. Rev. Lett*, 45(9):712, 1980.

[2] F. Takens. Detecting strange attractors in turbulence. In D.A. Rand and L.S. Young, editors, *Dynamical Systems and Turbulence*, volume 898 of *Lecture Notes in Mathematics*, pages 366–381. Springer, Berlin, 1981.

[3] T. Sauer, J.A. Yorke, and M. Casdagli. Embedology. *J. Stat. Phys.*, 65:95–116, 1991.

[4] P. Grassberger and I. Procaccia. Characterization of strange attractors. *Phys. Rev. Lett.*, 50:346–349, 1983.

[5] A. Wolf, J.B. Swift, H.L. Swinney, and J.A. Vastano. Determining Lyapunov exponents from a time series. *Physica D*, 16:285–317, 1985.

[6] R. van de Weygaert, B.J.T. Jones, and V.J. Martinez. The minimal spanning tree as an estimator for generalized dimensions. *Physics Letters*, 169A:145–150, 1992.

[7] R.F. Voss and J.C.Y. Wyatt. Multifractals and the local connected fractal dimension. In A.J. Crilly, R.A. Earnshaw, and H. Jones, editors, *Applications of Fractals and Chaos.* Springer-Verlag, Berlin, 1993.

[8] G. Mindlin and R. Gilmore. Topological analysis and synthesis of chaotic time series. *Physica D*, 58:229–242, 1992.

[9] N.B. Tufillaro, P. Wyckoff, R. Brown, T. Schreiber, and T. Molteno. Topological time-series analysis of a string experiment and its synchronized model. *Physical Review E*, 51(1):164–174, 1995.

[10] M.R. Muldoon, R.S. MacKay, J.P. Huke, and D.S. Broomhead. Topology from a time series. *Physica D*, 65:1–16, 1993.

[11] K. Mischaikow, M. Mrozek, A. Szymczak, and J. Reiss. From time series to symbolic dynamics: An algebraic topological approach. Preprint, December 1997.

[12] T. Kaczynski and M. Mrozek. Conley index for discrete multivalued dynamical systems. *Topology Appl.*, 65:83–96, 1995.

[13] T. Y. Kong and A. Rosenfeld. Digital topology: introduction and survey. *Computer Vision, Graphics and Image Processing*, 48:357–393, 1989.

[14] M. Brady, J.M. Hollerbach, T.L. Johnson, T. Lozano-Perez, and M. T. Mason, editors. *Robot Motion: Planning and Control.* MIT Press, 1984.

[15] J. Williams, editor. *Geographic information from space : processing and applications of geocoded satellite images.* Wiley, 1995.

[16] J.K. Udupa. Surface connectedness in digital spaces. In A. Rosenfeld and T.Y. Kong, editors, *Topological Algorithms for Digital Image Processing.* Elsevier, Amsterdam, 1993.

[17] G.T. Herman and E. Zhao. Jordan surfaces in simply connected digital spaces. *J. of Math. Imaging and Vision*, 6:121–138, 1996.

[18] A. Rosenfeld. Connectivity in digital pictures. *Journal of the ACM*, 17:146–160, 1970.

[19] T. Poston C.-N. Lee and A. Rosenfeld. Winding and Euler numbers for 2d and 3d digital images. *Computer Vision, Graphics and Image Processing: Graphical Models and Image Processing*, 53:522–537, 1991.

[20] T. Kong, A.W. Roscoe, and A. Rosenfeld. Concepts of digital topology. *Topology and its Applications*, 46:219–262, 1992.

[21] J.G. Hocking and G.S. Young. *Topology.* Addison-Wesley, 1961.

[22] H.-O. Peitgen, S. Jürgens, and D. Saupe. *Chaos and Fractals: New Frontiers of Science*. Springer-Verlag, 1992.

[23] K.J. Falconer. *Fractal Geometry: mathematical foundations and applications.* Wiley, New York, 1990.

[24] S. Newhouse. Diffeomorphisms with infinitely many sinks. *Topology*, 13:9–18, 1974.