Verifying Safety Properties of Hybrid Systems.

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Talk Outline

1. Formal Verification
2. Hybrid Systems
3. Invariant Synthesis
4. Algebraic Invariants
5. Results

Proving Programs Correct

Example: Compute $\lceil \sqrt{n} \rceil$, for $n \geq 0$.

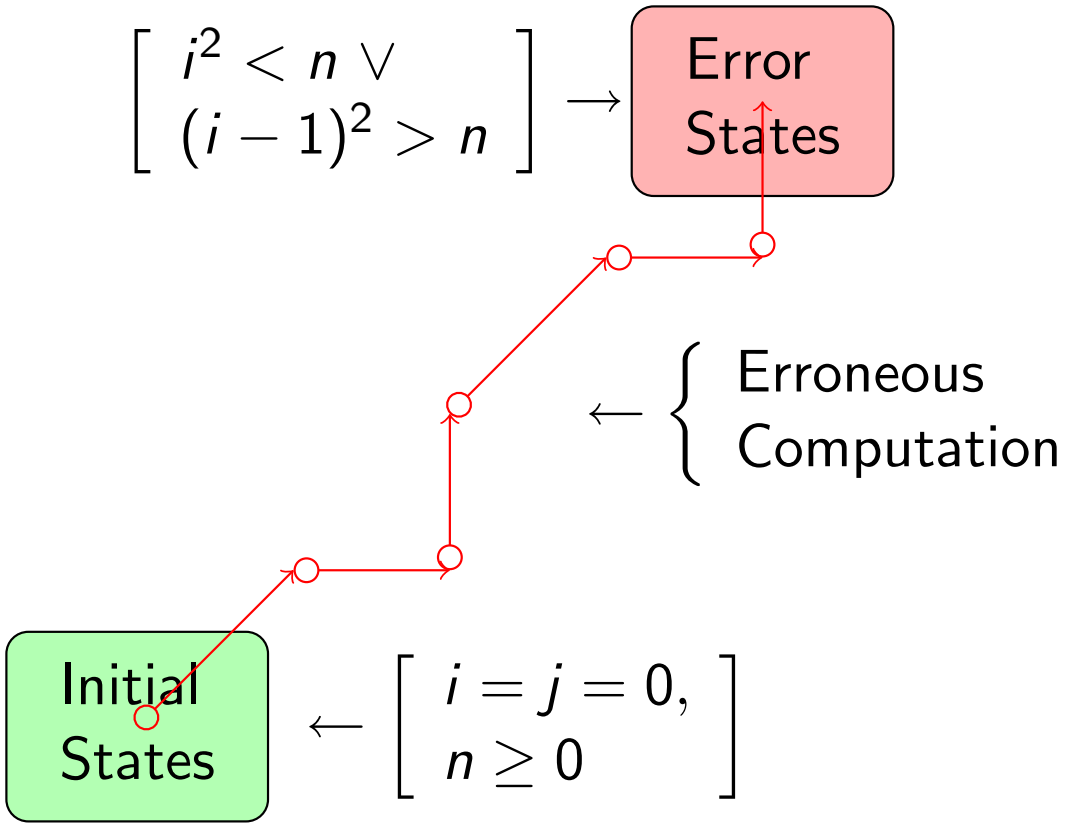
```
int computeSqrt ( int n )
      @Pre: n ≥ 0
1:    int i, j = (0, 0);
2:    while ( j ≤ n ) {
3:          i := i + 1;
4:          j := j + 2 * i − 1;
5:    }
      @Post: i^2 ≥ n ∧ (i − 1)^2 ≤ n
```

Verification Problem: Is this program "correct"?

What is Verification?

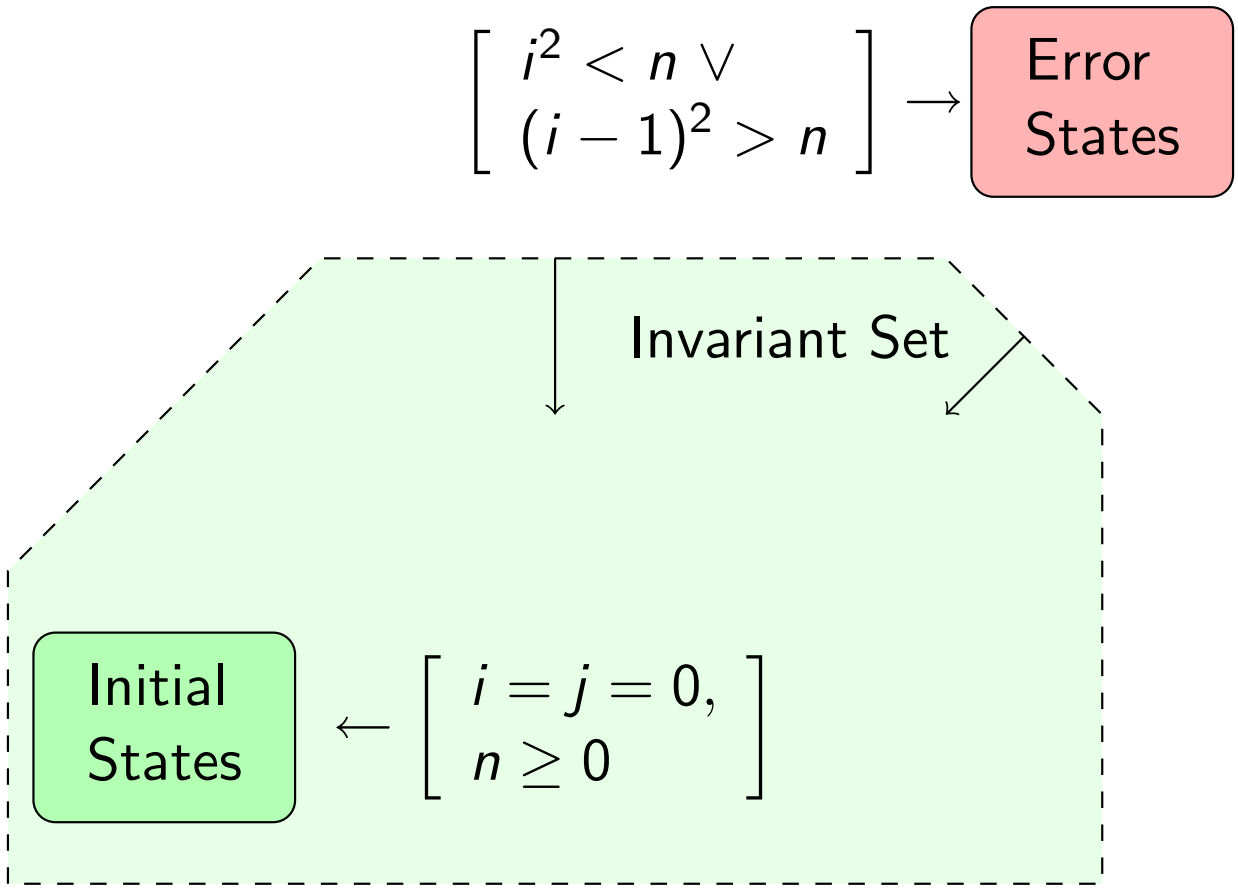
Finding an Error

Discover input n that violates property.

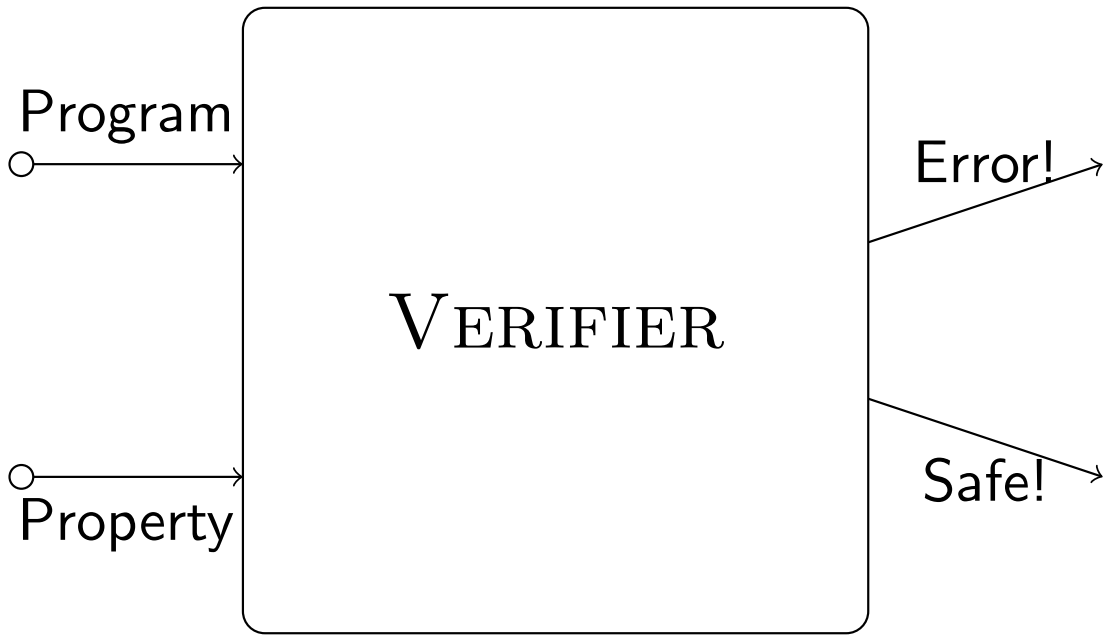


Proving Safety

Start inside invariant set $\Rightarrow$ remain in invariant set.



The Program Verifier



Undecidability

Theorem

There is no program P, that takes a program Q and a *non-trivial property* φ as inputs, and decides if every run of P satisfies φ.

Cf. Turing, Alan. M. (1936). "*On Computable Numbers, with an Application to the Entscheidungsproblem*". Proceedings of the London Mathematical Society. 2 42: 23065. 193637.

Verification Techniques

Nearly 40+ years of research on program verification.

- Model Checking: Best effort search for a counter example.
- Static Analysis: Best effort search for a property proof.
- Semi-Formal Methods: *Ad-hoc* approaches inspired by above.

Verification: Achievements

- Model Checking: Popular in hardware verification.
  - Arithmetic-Logic Unit Verification (Intel, AMD).
  - Cache Coherence Protocols.
  - Symbolic techniques (BDDs, SAT solvers): $\geq 10^{500}$ states.

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- Semi-Formal Tools: Finding bugs in programs.
  - CoVerity: Gaining wide usage in software industry.
  - Varvel (NEC), Slam (Microsoft), Findbugs (Google),...

Proving Programs Correct

Example: Compute $\lceil \sqrt{n} \rceil$, for $n \geq 0$.

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Verification Problem: Is this program "correct"?

Yes: Automatically synthesized invariant.

$$j = i^2,\ n \geq 0,\ j \leq n+1,\ j \leq n-2i-1\,.$$

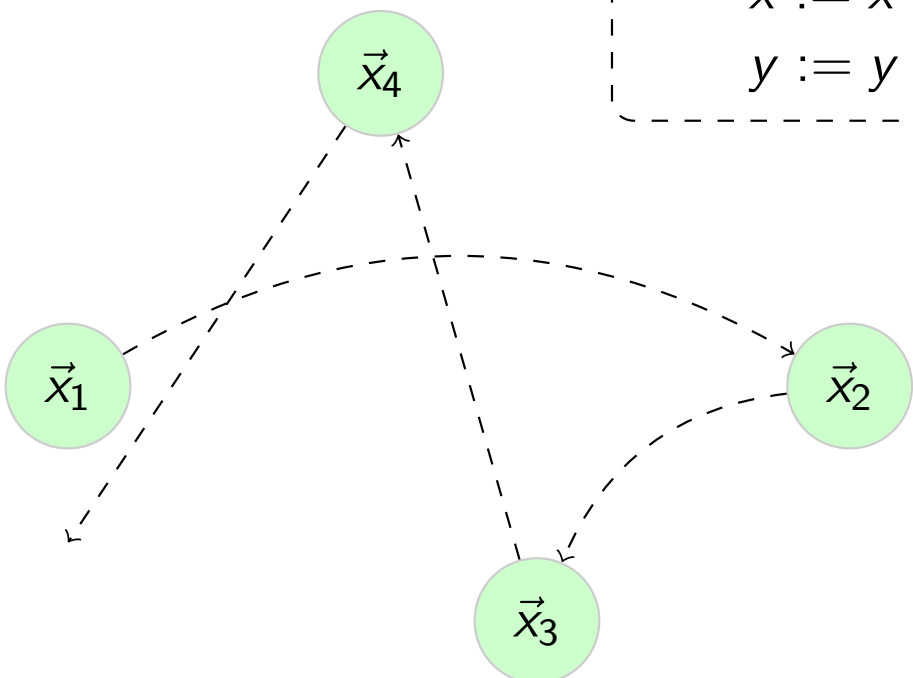
Challenge: Verify *Hybrid Systems*.

Dynamical Systems

Discrete dynamical systems: defined by maps.

$$\vec{x}(n+1) = F(n, \vec{x}(n))\,.$$

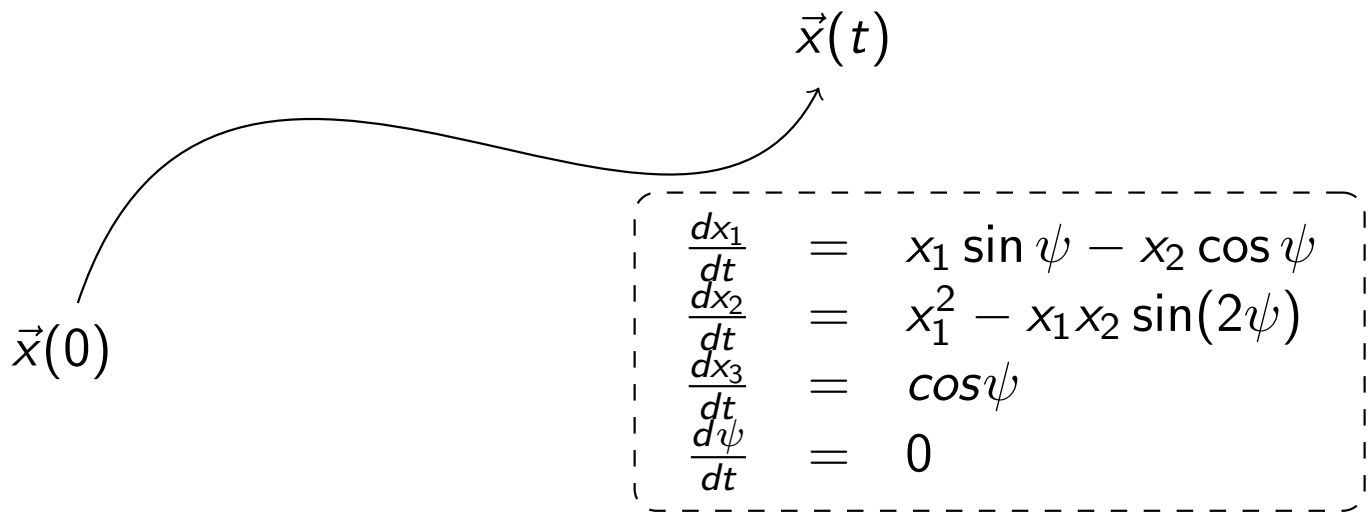
$$\begin{array}{c} \text{while}(x^2 + y^2 \leq 3) \\ x := x + y^{\frac{1}{2}} \\ y := y - x^{\frac{1}{2}} \end{array}$$



Dynamical Systems

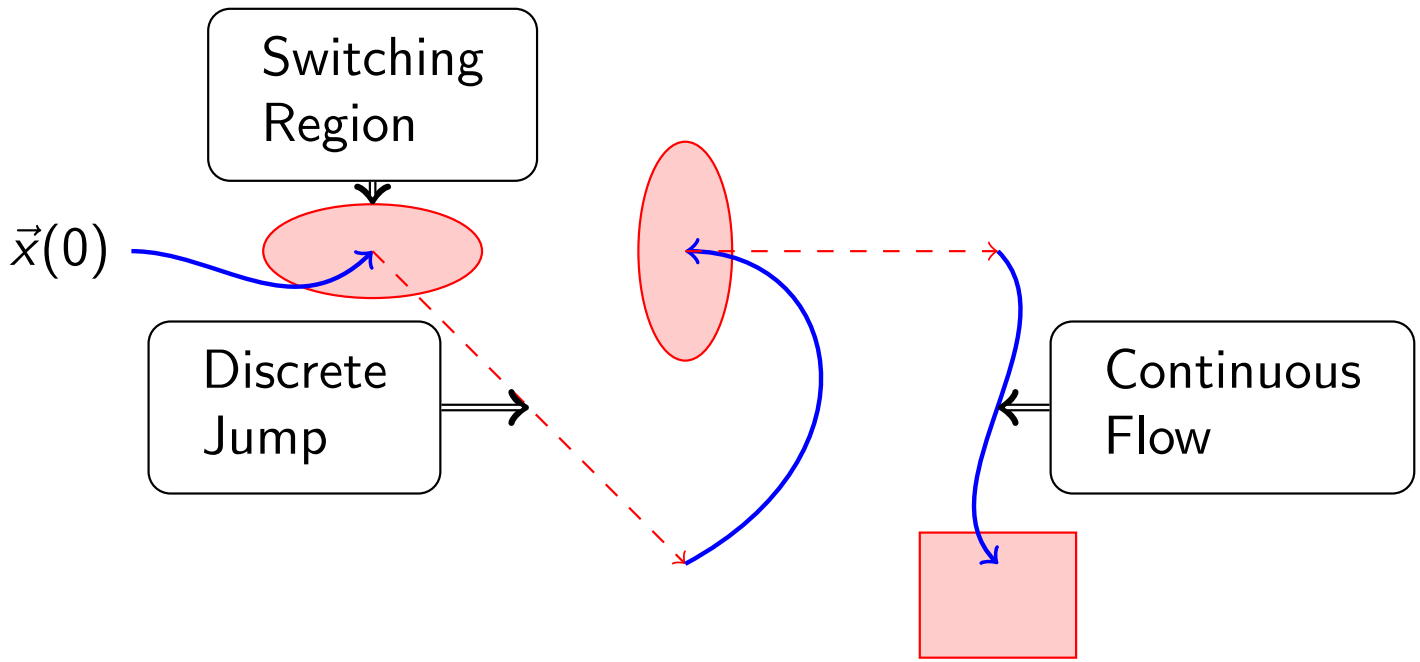
Continuous dynamical systems: defined by flows.

$$\frac{d\vec{x}}{dt} = F(t, \vec{x})\,.$$

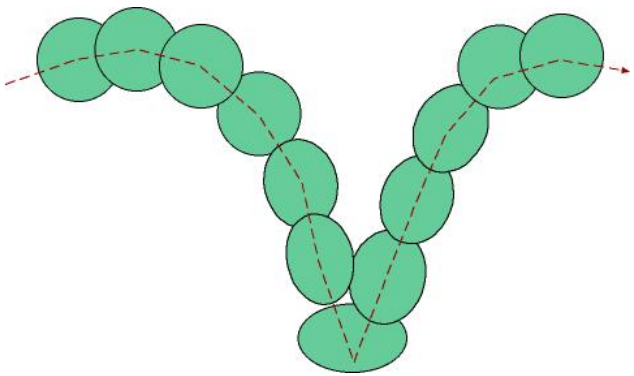


Hybrid Trajectories

1. Flows + Discrete jumps.
2. *Multi-Modal*: Dynamics depend on the mode.



Example # 1: Bouncing Ball

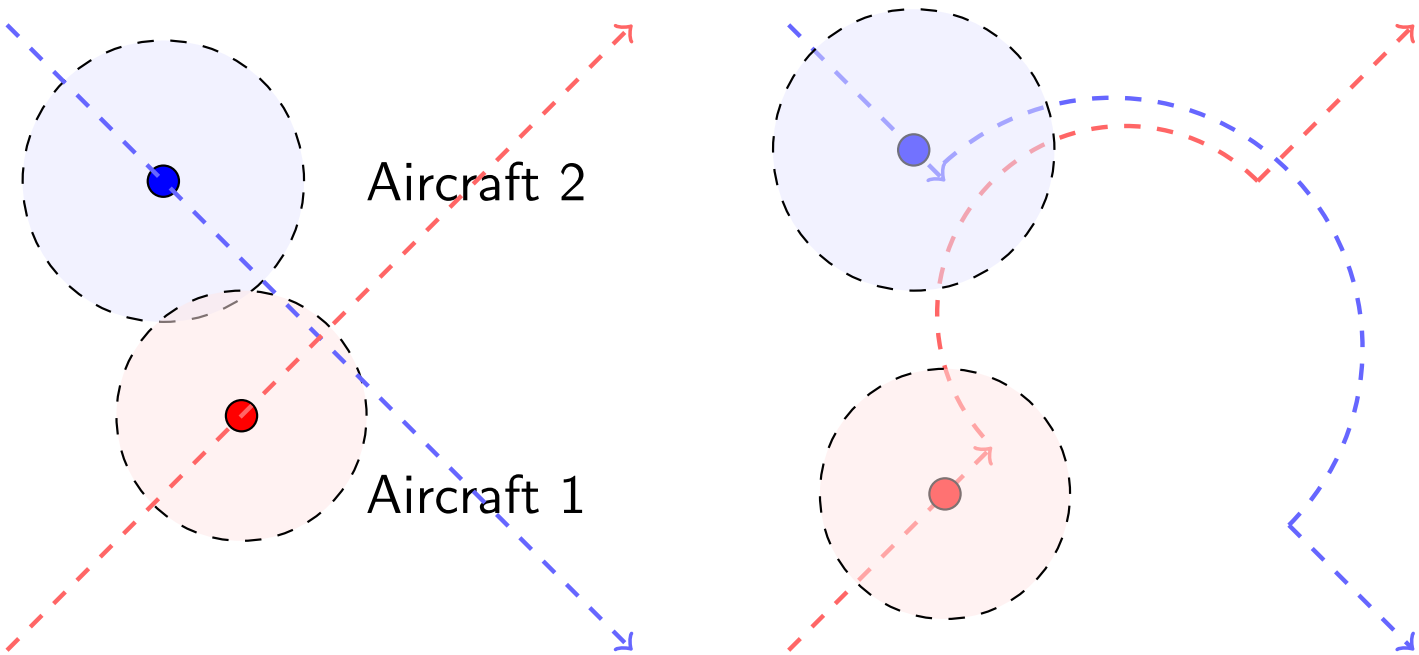


$$\text{if}(y = 0 \ \wedge \ v_y < 0)$$
$$\text{do} : v_y := -\tfrac{1}{2} v_y$$

$$\begin{aligned} \frac{dx}{dt} &= v_x \\ \frac{dy}{dt} &= v_y \\ \frac{dv_x}{dt} &= 0 \\ \frac{dv_y}{dt} &= -9.8 \end{aligned}$$

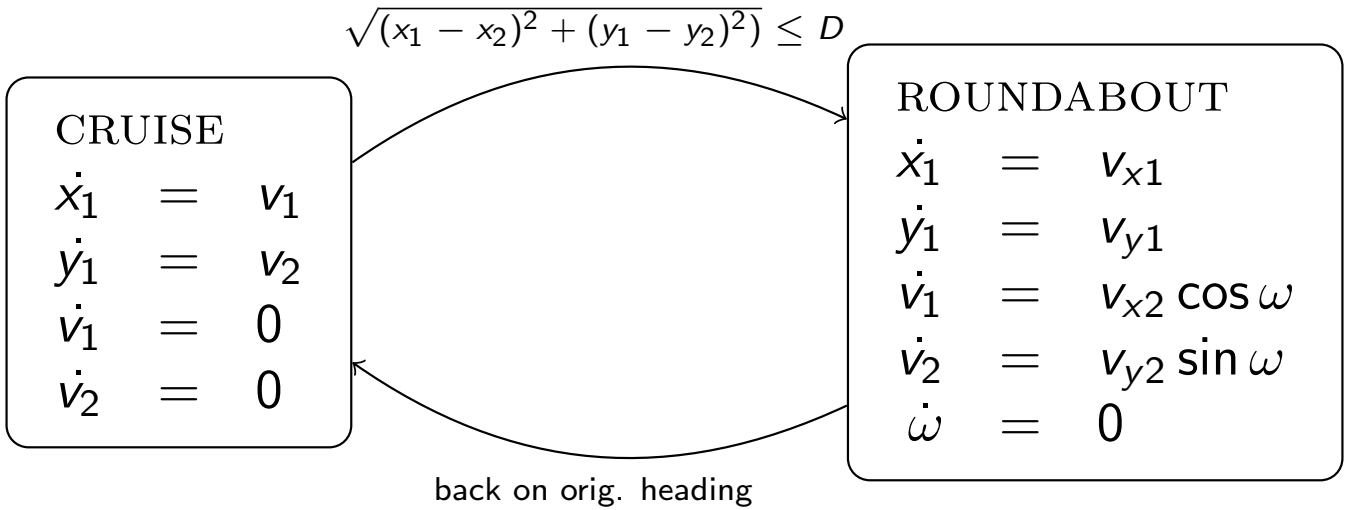
Example # 2: Conflict Resolution Maneuvers

Conflict resolution protocol. [Tomlin et al.'98]

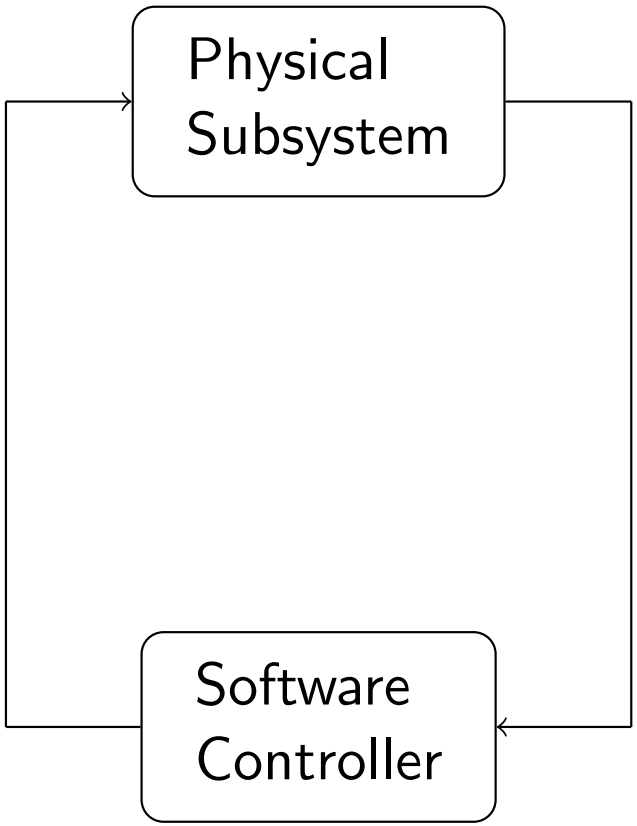


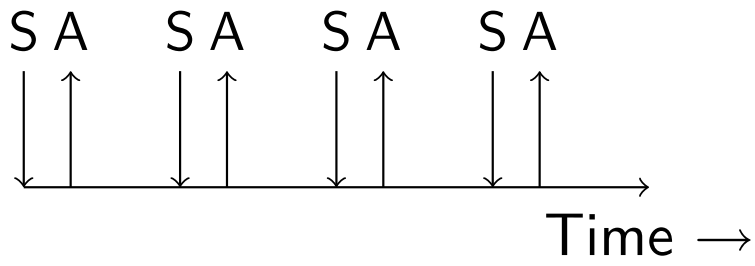
Collision Avoidance Model

Hybrid automaton for each aircraft:



Software Enabled Control

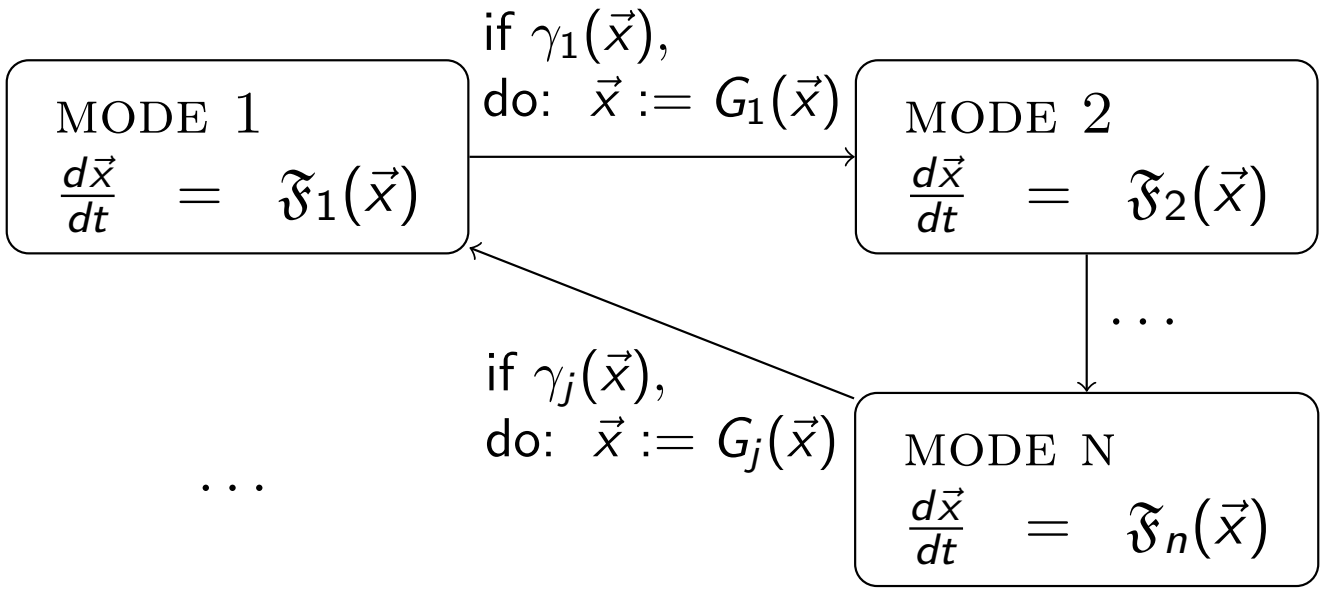




S: Sense, A: Actuate

Hybrid Automaton

[Alur et al.'96; Sastry et al. '98]



- Finite set of *modes*. $Q : \{q_1, \ldots, q_m\}$
- Continuous state variables. $\vec{x} : (x_1, \ldots, x_n)$.
- Dynamics for each mode.
- Discrete Transitions between modes.

Verification of Hybrid Systems.

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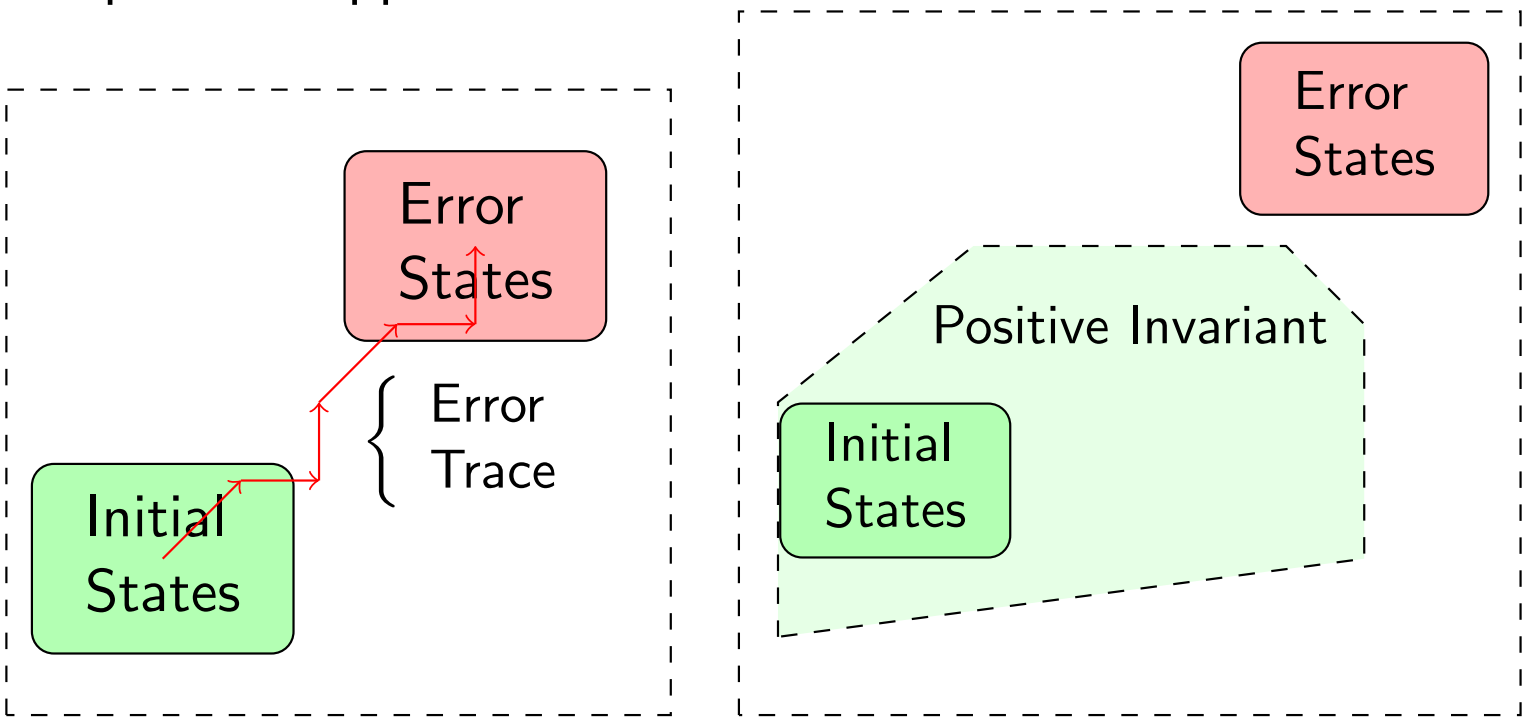
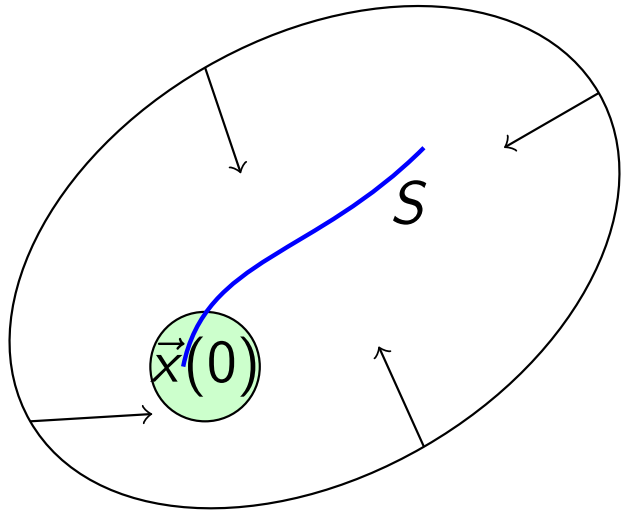
Two possible approaches.



Rest of this talk: Generating Positive Invariants.

Positive Invariant Set

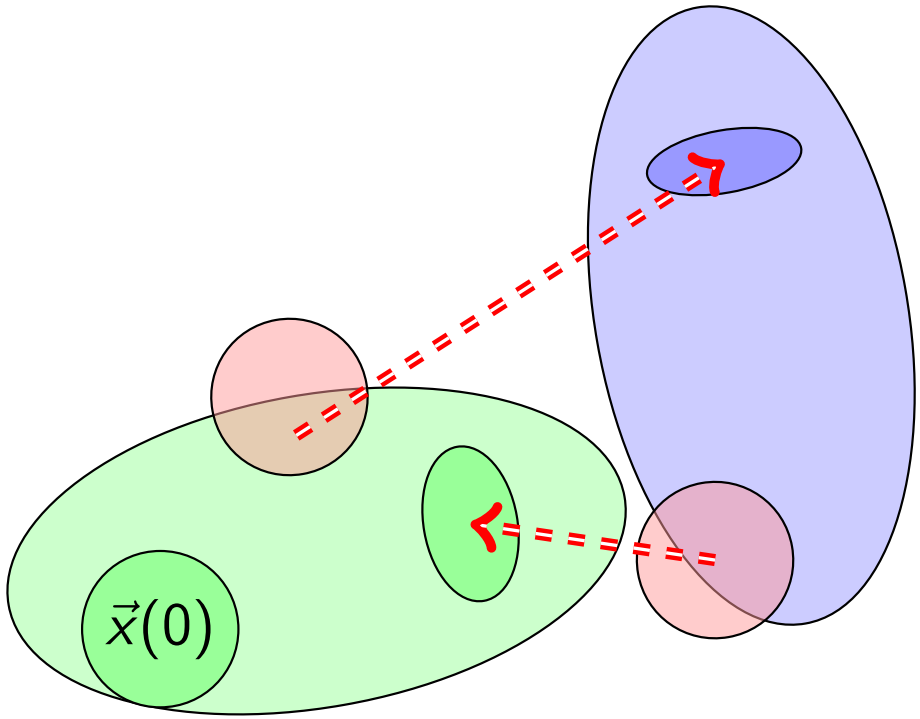


Set S is positive invariant for flow φ iff

$$\forall\ \vec{x}(0) \in S,\ \varphi(\vec{x}(0), t) \in S\,.$$

Start inside set $S \Rightarrow$ flow remains in S.

Positive Invariants for Hybrid Systems



- Disjoint union of sets.
- "Preserved" by discrete transitions.

Positive Invariant Computation.

[Henzinger et al.'96,...]

$\vec{x}(0)$

- Start with Initial States.

Positive Invariant Computation.

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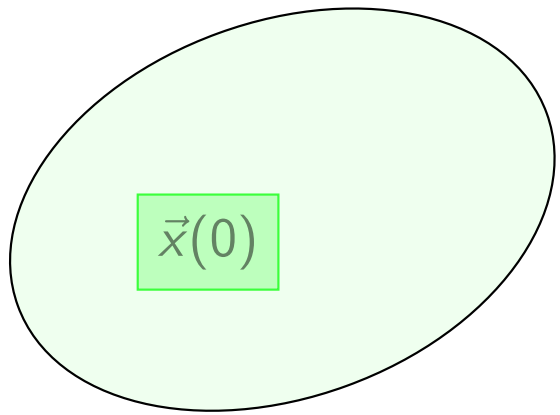
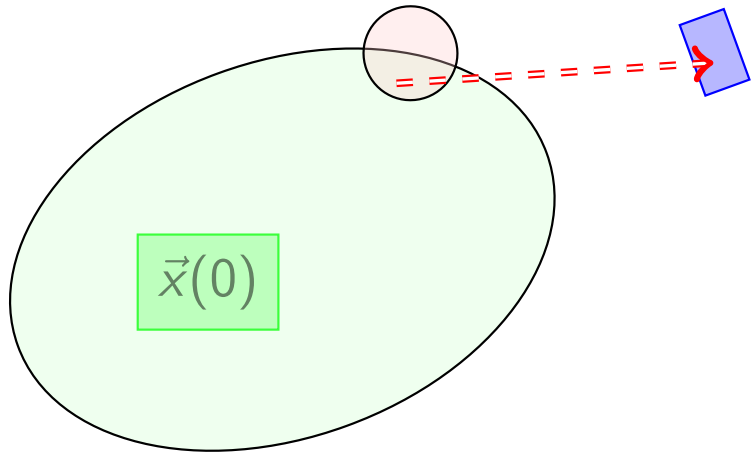


- Start with Initial States.
- Positive Invariant computation for differential equations.

Positive Invariant Computation.

[Henzinger et al.'96,...]



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Positive Invariant Computation.

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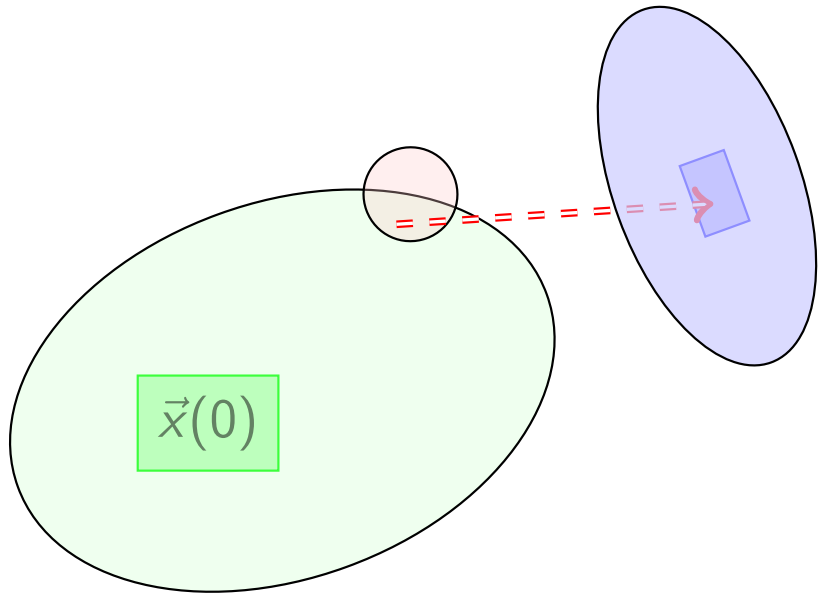

- Start with Initial States.
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- Computing images across discrete transitions.
- Iterate until convergence.

Positive Invariant Computation.

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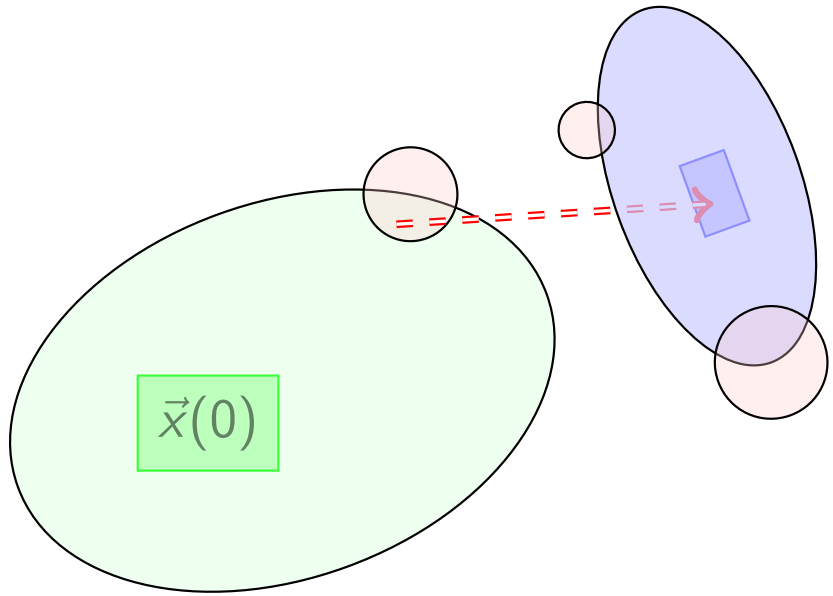

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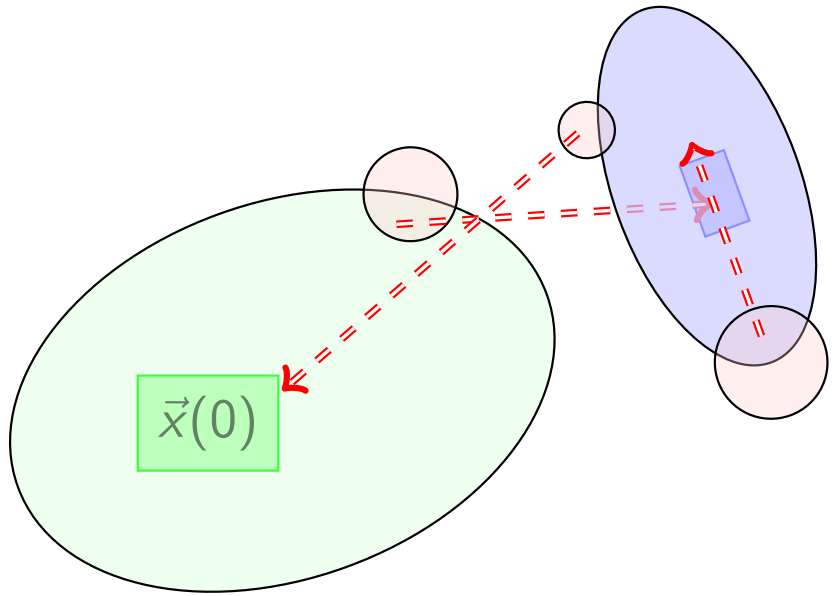

- Start with Initial States.
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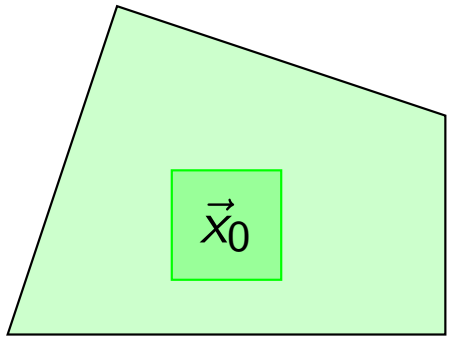
Polyhedral Invariant Generation.



- Polyhedra [Halbwachs'94, Henzinger et al.'96, Sank.et al.'06]
- Zonotopes. [Girard'05]
- (Linear) Support Functions. [Girard'09,'10]
- Templatized Polyhedra. [Sank.et al.'08,'09]
- Orthogonal Polyhedra. [Asarin et al.'02]

- Tools: PHaver, Mattisse, TimePass,...

Q: What about non-linear invariants and non-linear dynamics?

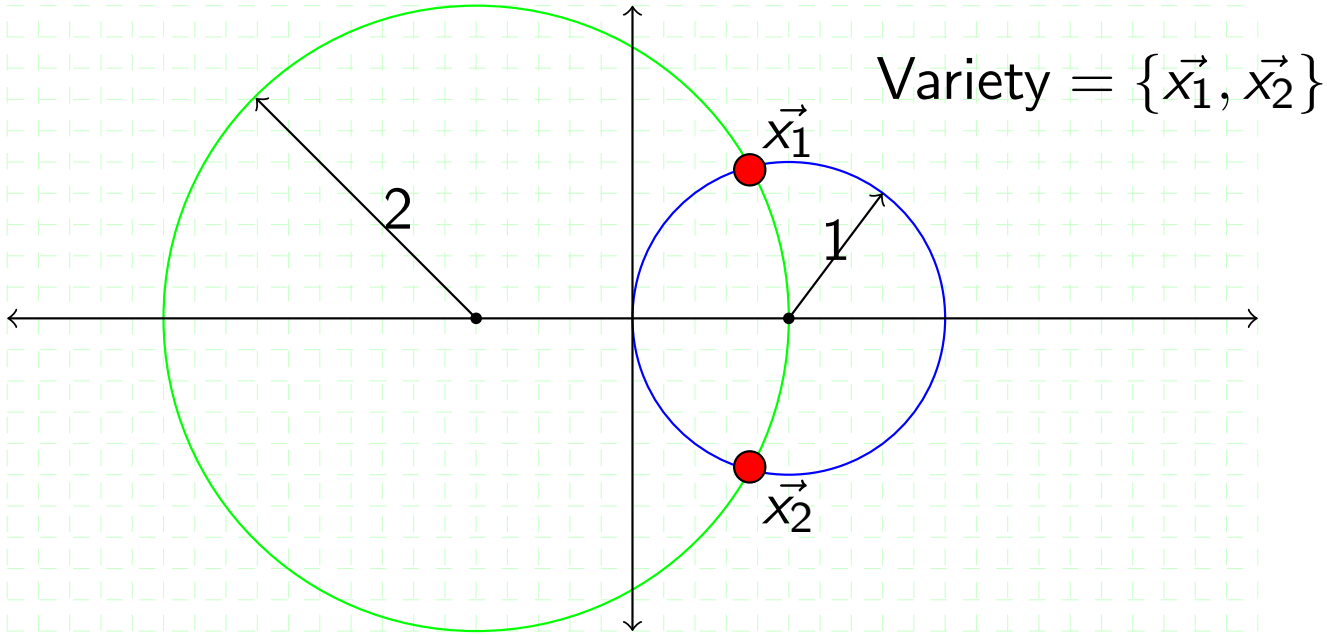
Algebraic Techniques for Non-Linear Invariants.

- Generate positive invariants that are *algebraic varieties*.

$$x_1^2 - 2x_3 - x_2^2 = 5 \ \wedge \ x_1 x_2 = x_3$$

- Using techniques from *computational algebraic geometry*.

Algebraic Varieties



Zeros of a set of multi-variate polynomials.

$$V : \{\vec{x} \in \mathcal{R}^n \mid p_1(\vec{x}) = 0,\ p_2(\vec{x}) = 0,\ \ldots,\ p_m(\vec{x}) = 0\}$$

Ideals

Set of polynomials I closed under

1. Addition.

$$p_1, p_2 \in I \;\Rightarrow\; p_1 + p_2 \in I$$

2. Multiplication by any other polynomial in the ring.

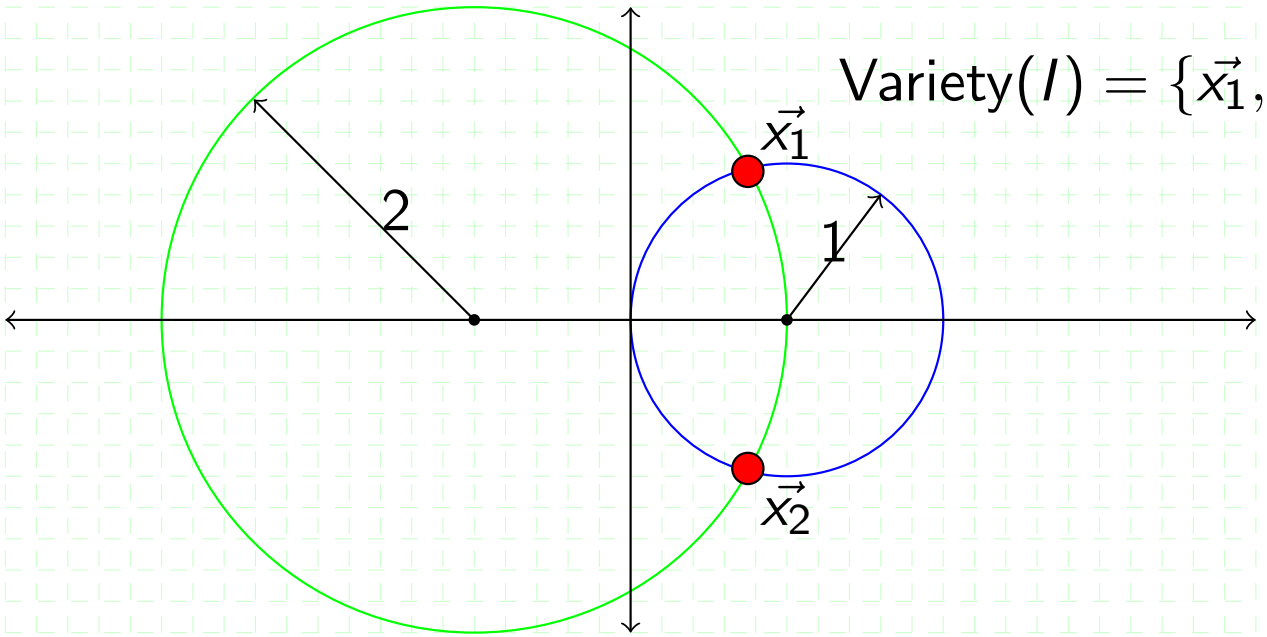
$$p \in I,\; g \in \mathcal{R}[\vec{x}],\; \Rightarrow\; g \cdot p \in I$$

Ideal Generated by Set: $\langle\langle p_1, \ldots, p_m \rangle\rangle$.

Background: Ideals and Varieties.

$$\begin{aligned} \text{IdealOf}(V) &: \{p \in K[\vec{x}] \mid p \text{ vanishes everywhere on } V\} \\ \text{VarietyOf}(I) &: \{\vec{x} \in K^n \mid \forall\ p \in I,\ p(\vec{x}) = 0\} \end{aligned}$$

Example: $I = \left\langle\left\langle \{\underbrace{(x+1)^2 + y^2 - 4}_{p_1}, \underbrace{(x-1)^2 + y^2 - 1}_{p_2}\} \right\rangle\right\rangle.$



Hilbert's Nullstellensatz

For any ideal I and polynomial p,

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(i.e, p is a consequence of the equations in I)

if and only if there is some power p^m of p, such that

$$p^m \in I$$

Computing with Ideals

- Hilbert Finite Basis Theorem:

  Any ideal in $K[\vec{x}]$ for field K is finitely generated.

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  Any ideal in $K[\vec{x}]$ for field K is finitely generated.

- *Groebner Basis:* Finite basis with nice properties.
  [Buchberger'60]

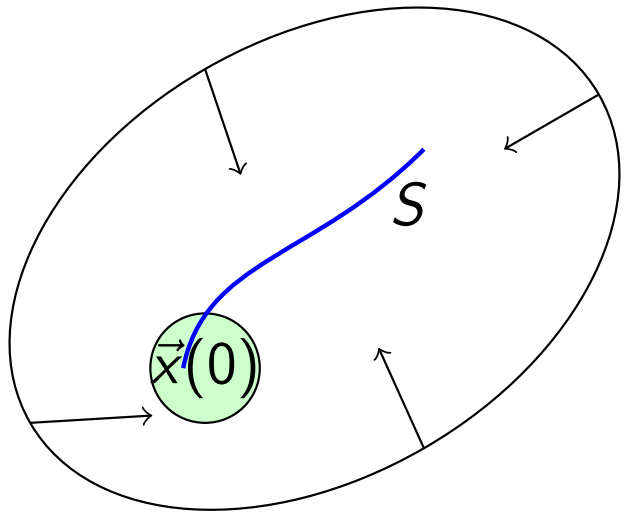
- Computing using Varieties.
  1. Ideal Membership (subsumption of varieties)
  2. Ideal Intersection (union of varieties)
  3. Ideal Union (intersection of varieties)
  4. Ideal Inclusion (inclusion of varieties)
  5. Image under a Map
  6. *Syzygy Modules*
  7. ...

Positive Algebraic Invariants.

Invariant synthesis for flows.

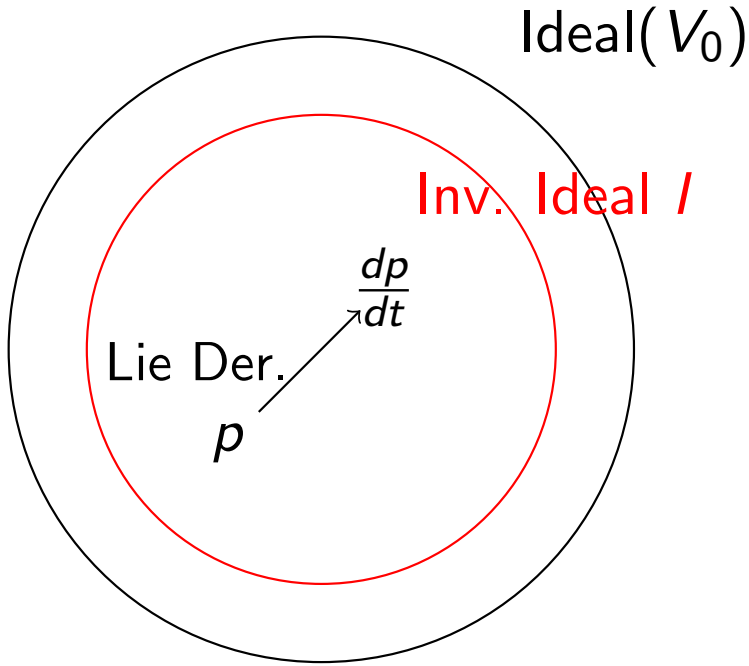
- Inputs:
  (a) Polynomial vector field: $\mathfrak{F}$.
  (b) Initial Set (variety): V_0.
- Problem: Generate positive invariant algebraic variety.
- "Smaller" variety is more desirable/useful.

Positive Invariant Variety



1. Contain initial variety
2. Vector field must lie on the tangent space.

"Positive Invariant" Ideal.



1. Contained in initial ideal: Ideal(V_0).
2. Closed under Lie-Derivatives w.r.t vector field.

Problem: Given V_0 and field $\mathfrak{F}$, compute *invariant ideal*.

Example # 1: Mechanical System

$$\begin{aligned}
\frac{dp_1}{dt} &= -2q_1q_2^2, \\
\frac{dp_2}{dt} &= -2q_1^2q_2, \\
\frac{dq_1}{dt} &= 2p_1, \\
\frac{dq_2}{dt} &= 2p_2
\end{aligned}$$

Initial condition:

$$V_0 : \{\vec{x} : (p_1, q_1, p_2, q_2) \mid p_1^2 + p_2^2 - 4 = 0,\ q^2 - 1 = 0, q_2 = 0\}.$$

Positive Invariant: $p_1^2 + p_2^2 + q_1^2q_2^2 = 4$.

A More Complex Example

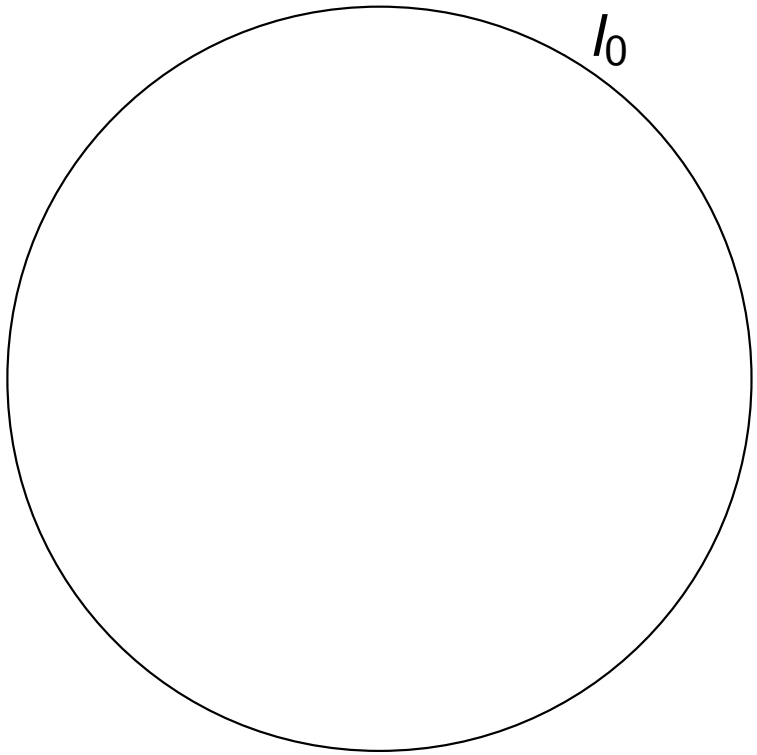
$$\begin{array}{rcl} \frac{dx_1}{dt} & = & v_1, \\ \frac{dx_2}{dt} & = & v_2, \\ \frac{dv_1}{dt} & = & -kx_1 - \frac{k}{5}(x_1 - x_2), \\ \frac{dv_2}{dt} & = & k(x_1 - x_2), \\ \frac{dk}{dt} & = & 0 \end{array}$$

$$V_0 : x_1 = x_2 = 0, v_1 = 1, v_2 = -1, k \in R$$

Invariant Ideal:

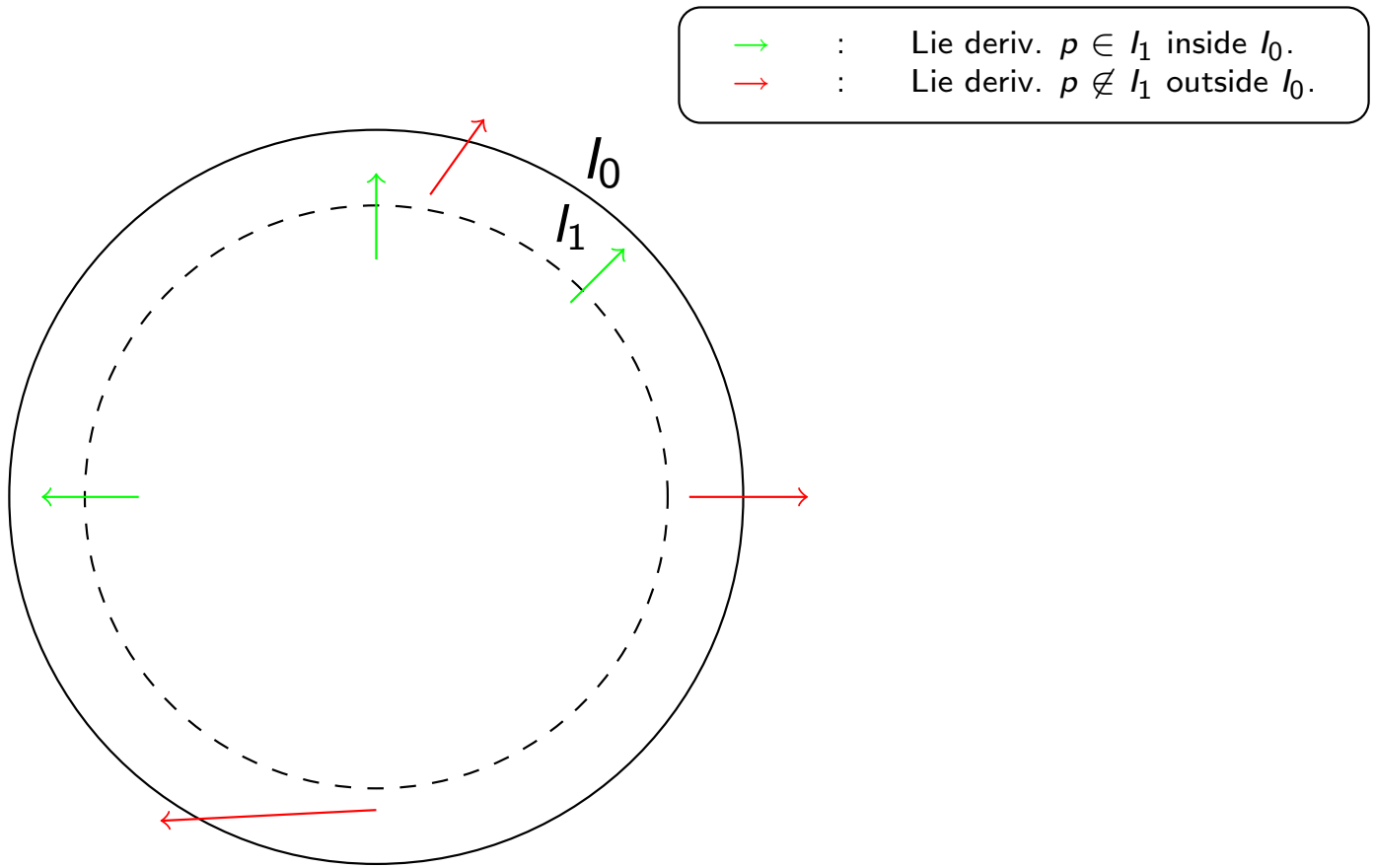
$$\begin{array}{lcl} p_1 & : & 576 + 1200v_1^2 + 625v_1^4 + 2880v_1v_2 + 3000v_1^3v_2 + \\ & & 528v_2^2 + 4150v_1^2v_2^2 + 1320v_1v_2^3 + 121v_2^4 \\ & & -1860kx_2^2 + 2750kv_1^2x_2^2 + 1600kv_1v_2x_2^2 \\ & & +710kv_2^2x_2^2 + 525k^2x_2^4 = 0, \\ p_2 & : & 240x_1 + 250v_1^2x_1 + 600v_1v_2x_1 + 110v_2^2x_1 \\ & & +396x_2 - 525v_1^2x_2 - 260v_1v_2x_2 - 131v_2^2x_2 - 105kx_2^3 = 0, \\ p_3 & : & 24 + 25v_1^2 + 60v_1v_2 + 11v_2^2 + 50kx_1x_2 + 5kx_2^2 = 0 \\ p_4 & & -21 + 25v_1^2 + 10v_1v_2 + 6v_2^2 + 25kx_1^2 + 5kx_2^2 = 0 \end{array}$$

Strategy



1. Start with initial ideal I_0 and iterate.
2. Iterative Step: $I_{j+1} = I_j \cap \{p \mid \mathrm{Lie}_F(p) \in I_j\}$.
3. Stop if $I_n = I_{n+1}$.

Strategy



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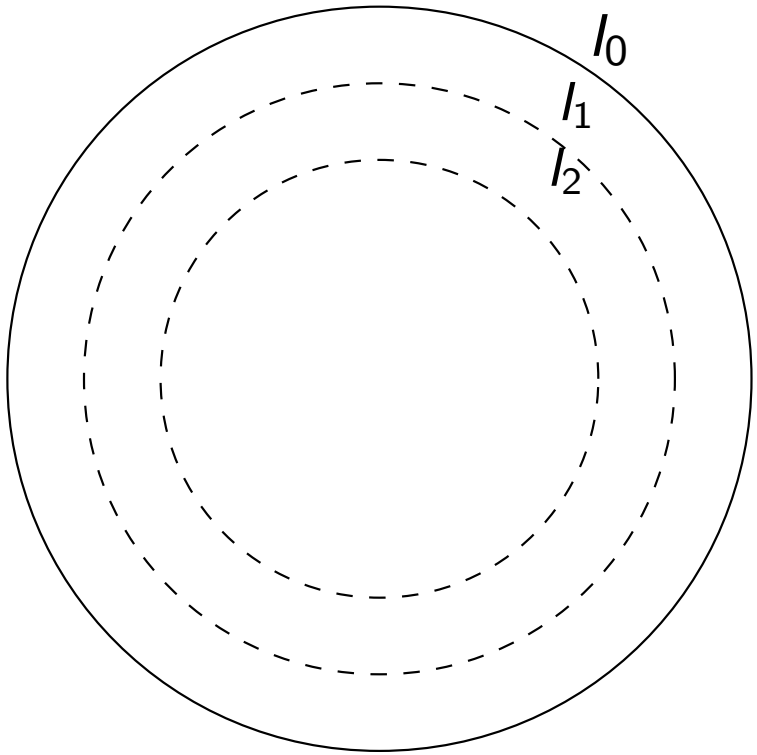

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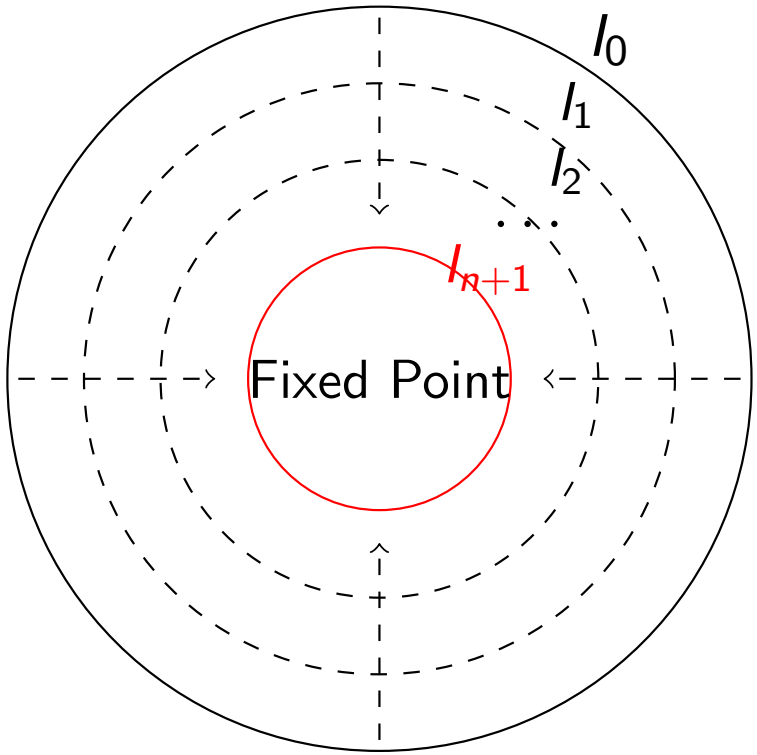

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Ideal Refinement

$$\partial_F I = I \cap \{p \mid \mathsf{Lie}_F(p) \in I\}\,.$$

I.e, retain $p \in I$ such that $\mathcal{L}_F(p) \in I$.

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Algorithm:

1. *Compute Groebner basis* G of I.
2. *Compute Lie derivative* of each generator G.
3. *Intersect ideal* generated by Lie derivatives with I.
4. *Compute Syzygies* for each generator of the intersection.
5. *Multiply matrix* representing syzygies with generators in G.
6. Result is the set of generators of $\partial_F I$.

Convergence

- Convergence if the ring $K[\vec{x}]$ satisfies descending chain condition.
- But ideals in $\mathcal{R}[\vec{x}]$ do not satisfy *descending chain condition*.
- Solution: "Under-approximate" iteration inside a vector space.

Pseudo-Ideal

Finite dimensional vector-space inside an ideal. [Colón, 2004]

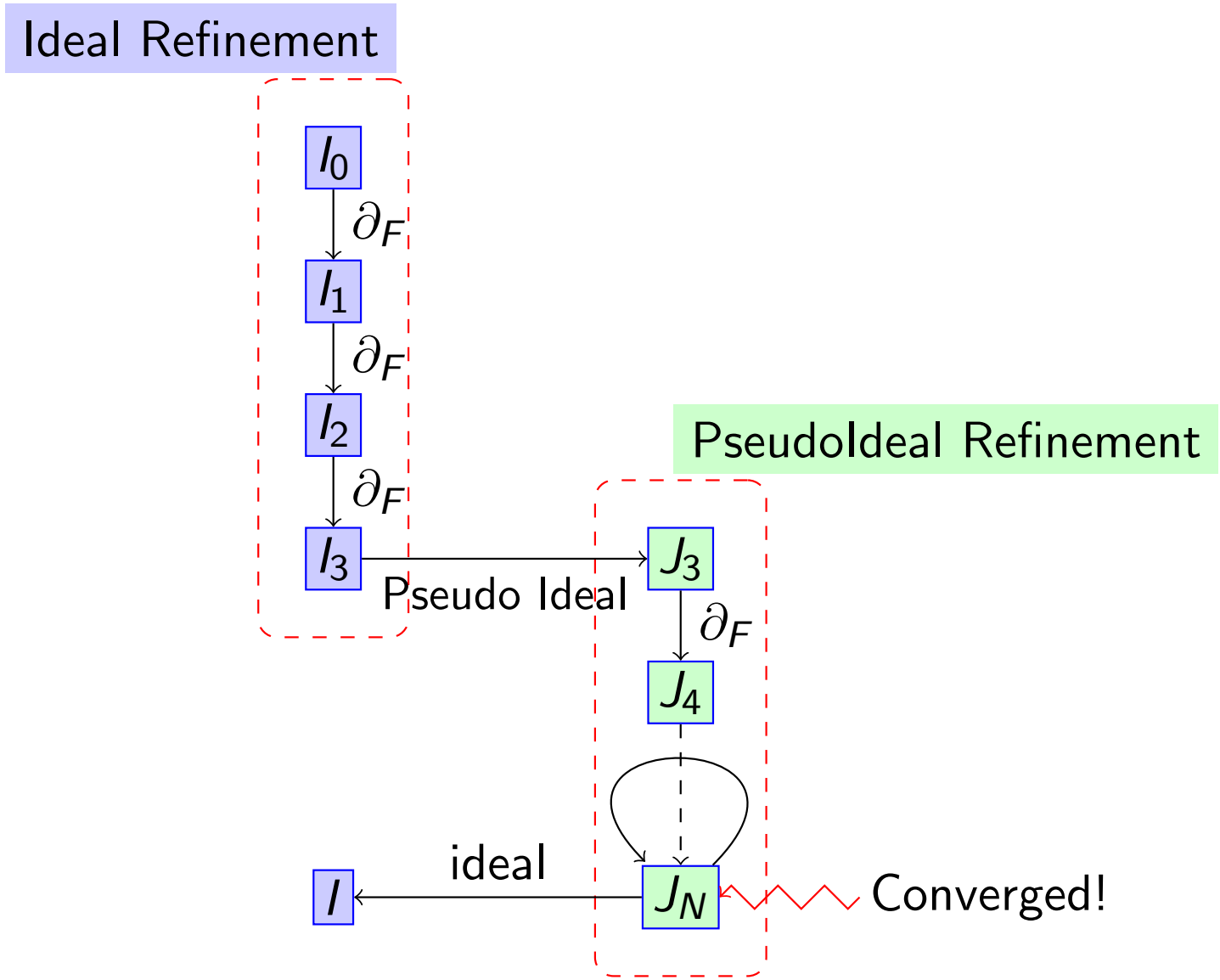
- Closure under addition.
- Multiplication closure with *polynomials with degree* $\leq d$.
- Vector space of polynomials.
- Descending chain condition holds.

Further Details

Sank., Automatic Invariant Generation for Algebraic Systems using Ideal Fixed Points, Hybrid Systems: Computation and Control, Apr. 2010

Implementation

Combine ideal and pseudo-ideal iterations:



Aircraft Collision Avoidance

Vector Field:

$$\begin{array}{cccc} x_1' = d_1 & x_2' = d_2 & d_1' = -\omega d_2 & d_2' = \omega d_1 \\ y_1' = e_1 & y_2' = e_2 & e_1' = -\theta e_2 & e_2' = \theta e_1 \\ a' = 0 & b' = 0 & r_1' = 0 & r_2' = 0 \end{array}$$

Initial set

$$\left[\begin{array}{c} x_1 = y_1 = r_1 \ \wedge \ x_2 = y_2 = r_2 \ \wedge \ d_1 = a \ \wedge \\ d_2 = 0 \ \wedge \ e_1 = b \ \wedge \ e_2 = 0 \end{array} \right]$$

Positive Invariant Set Obtained:

$$
\begin{aligned}
&\mathbf{p_1}: e_1^2 + e_2^2 - b^2, \\
&\mathbf{p_2}: d_1^2 + d_2^2 - a^2 \\
&\mathbf{p_3}: e_1 - r_2\theta + \theta y_2, \\
&\mathbf{p_4}: -a + d_1 - r_2\omega + \omega x_2 \\
&\mathbf{p_5}: b - e_2 - r_1\theta + \theta y_1, \\
&\mathbf{p_6}: -br_1 + by_1 + e_1 r_2 - e_1 y_2 - e_2 r_1 + e_2 y_1 \\
&\mathbf{p_7}: br_2 - by_2 - e_1 r_1 + e_1 y_1 - e_2 r_2 + e_2 y_2 \\
&\mathbf{p_8}: -d_2 - r_1\omega + \omega x_1 \\
&\mathbf{p_9}: ad_2 r_2 - ad_2 x_2 + d_1 d_2 r_2 - d_1 d_2 x_2 - r_1 d_2^2 r_1 + d_2^2 x_1 \\
&\mathbf{p_{10}}: ar_1 - ax_1 - d_1 r_1 + d_1 x_1 - d_2 r_2 + d_2 x_2
\end{aligned}
$$

Ongoing Work

- Polynomial inequality invariants:
  - Refinement over cones of positive semi-definite (psd) polynomials.
  - Using *sum-of-squares* relaxation and semi-definite programming. [Shor'87, Parillo'03]

Ongoing Work

- Polynomial inequality invariants:
  - Refinement over cones of positive semi-definite (psd) polynomials.
  - Using *sum-of-squares* relaxation and semi-definite programming. [Shor'87, Parillo'03]
- Study *homomorphisms* between dynamical systems.
  - Similar *topological semi-conjugacy*.
  - Algorithm for computing *linearizing homomorphisms*.
  - Encouraging results. [Sank.(draft) '10]