Verification and Control of CPS: Assignment 5
Due Date: Tuesday, Nov 4, 2014 (4:30 PM)

P1 Consider the third order differential equation below:

$$\frac{d^3y}{dt^3} + 1.2\frac{d^2y}{dt^2} - 2.7\frac{dy}{dt} + 15.1y = 0$$

(a) Write it as a system of coupled ODEs by introducing new variables $w = \frac{d^2y}{dt^2}$, and $x = \frac{dy}{dt}$.

(b) The resulting coupled ODE is a linear system:

$$\frac{d}{dt}\begin{pmatrix} w \\ x \\ y \end{pmatrix} = A \begin{pmatrix} w \\ x \\ y \end{pmatrix} + \vec{b}$$

What are the matrices $A, \vec{b}$?

(c) Calculate the matrix exponentials e^{A}, e^{2A} and e^{3A}.

(d) For initial conditions given by $w = 0.2, x = -0.1$, and $y = 1$, write down the value of y at times $t = 1, 2$ and 3 second.

(e) Check whether the system is stable by computing its eigenvalues.

Note: You are free to use MATLAB (tm), Octave or Python to compute matrix exponentials and/or the eigenvalues. The Matlab/Octave function for matrix exponential is `expm` (and not `exp`).

(a) The coupled ODEs are

$$\begin{aligned} \frac{dw}{dt} &= -1.2w + 2.7x - 15.1y \\ \frac{dx}{dt} &= w \\ \frac{dy}{dt} &= x \end{aligned}$$

(b) The matrices are

$$A = \begin{bmatrix} -1.2 & 2.7 & -15.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Using matlab we obtain

```
  >> expm(A)

ans =

   -0.5359   -3.8605   -6.5257
    0.4322   -0.0173   -5.0274
    0.3329    0.8317   -0.9162


>> expm(2*A)

ans =
```

```
   -3.5539   -3.2922   28.8841
   -1.9129   -5.8494    1.8725
   -0.1240   -2.0617   -5.5145

>> expm(3*A)

ans =

   10.0983   37.7997   13.2798
   -0.8795    9.0429   40.1743
   -2.6605   -4.0721   16.2264

>>
```

(c) Once again, we use matlab to compute the states of the system, to obtain $y(1) = -0.9328,\ y(2) = -5.3332,\ y(3) = 16.1015$.

```
>> p0 = [ 0.2; -0.1; 1]

p0 =

    0.2000
   -0.1000
    1.0000

>> expm(A) *p0

ans =

   -6.2469
   -4.9392
   -0.9328

>> expm(2*A) *p0

ans =

   28.5025
    2.0749
   -5.3332

>> expm(3*A) *p0

ans =

   11.5195
   39.0941
   16.1015
```

>>

(d) The eigenvalues of A are $-3.3507, 1.0754 + 1.8303i, 1.0754 - 1.8303i$. Two eigenvalues lie on the right half of the complex plane and thus, the system is unstable.

P2 Consider the Vanderpol oscillator given by the system of coupled ODEs:

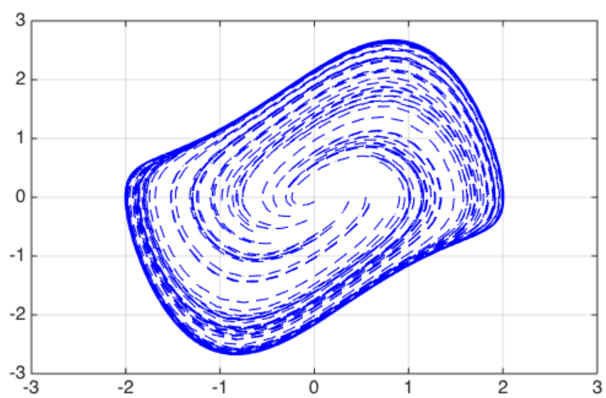
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x \end{aligned}$$

1. Find all the possible equilibria of this system. (**Hint**: set the RHS of the ODE to 0 and solve for x, y).
2. Using a Runge-Kutta solver (`ode23` function in MATLAB and equivalents) solve the ODE for various initial conditions randomly chosen inside the box $x \in [-1, 1]$ and $y \in [-1, 1]$ for time $t \in [0, 100]$ units. Plot the resulting trajectories.
3. Use the trajectories to decide if the system is *stable* or *unstable* at each of the equilibria found.
4. Draw a Simulink diagram for the Vanderpol system. Allow the simulation to set various values for $x(0), y(0)$ and be able to plot the result.

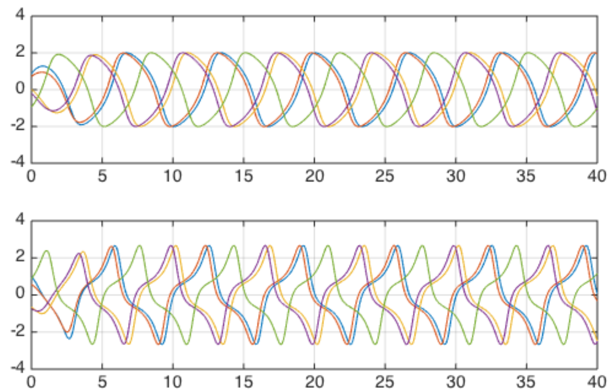
1. The equilibriua of the system are found by setting the RHS of the ODEs to 0.

$$y = 0,\ (1 - x^2)y - x = 0$$

Solving, we obtain the only equilibrium as $x = y = 0$.
2. Attached file `vdpSimulation.m` generates the trajectories shown below.



3. The system is unstable.
4. See the attached diagram `vdpModel.slx` and script `simulateVdpModel.m` that runs this model for various initial conditions for the integrators. The diagram below shows time plots for $x(t)$ and $y(t)$ for various initial values through simulating the Simulink diagram.



P3 Consider the following control system:

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u}\,,$$

wherein $A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix}$ with u as a single input.

1. Find a static state feedback stabilizing control law of the form $u = K\vec{x}$ using the idea of placing "eigenvalues" of the state space system as shown in the class. For this problem, the eigenvalues of the closed loop should be at $\lambda_1 = -1, \lambda_2 = -2$.

2. Implement a PID controller that will attempt to stabilize the value of x_1 to a reference point $x_1 = 3$. For this controller, do not use the values of x_2 in your feedback loop. Report the values of the various gains for this controller and show a plot of the closed loop simulation in Simulink.

1. With a feedback law $u = K\vec{x}$, the closed loop dynamics become

$$\frac{d\vec{x}}{dt} = (A + BK)\vec{x}\,.$$

Writing $K = [k_1 \;\; k_2]$, we obtain:

$$A + BK = \begin{pmatrix} 1 + k_1 & -1 + k_2 \\ 1 + 0.2k_1 & -2 + 0.2k_2 \end{pmatrix}$$

We wish the eigenvalues to be $\lambda = -1, -2$. Therefore, we equate the sum of eigenvalues to the trace of the matrix.

$$1 + k_1 - 2 + 0.2k_2 = -1 + -2 = -3, \text{ simplifies to } k_1 = -2 - 0.2k_2\,.$$

Likewise, we have the determinant to be the product of eigen values.

$$(1 + k_1)(-2 + 0.2k_2) - (1 + 0.2k_1)(-1 + k_2) = -2 * -1 = 2$$

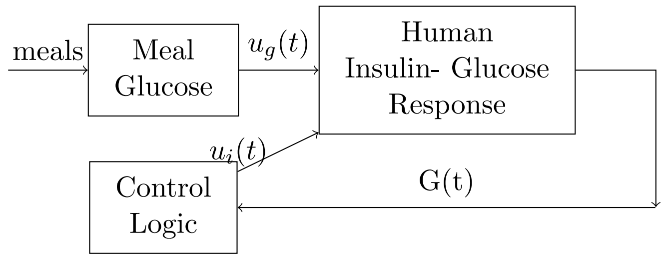
Simplifying, we obtain

$$1.8k_1 + 0.8 = -3$$

Solving, we obtain $k_1 = -2.2727$ and $k_2 = 1.3636$.

2. Simulink model is attached `prob3assign5.slx`.

P4 In this assignment you will model the different pieces of a simple artificial pancreas setup that controls blood glucose levels in people with type-1 diabetes.



The human insulin-glucose response is modeled by the Bergman minimal model with three state variables (G, I, X) wherein G is plasma glucose , concentration above the basal value G_B (units: mmol/L), and I is the plasma insulin concentration above the basal value I_B (units: U/L). X is the insulin concentration in an *interstitial chamber*. Note that time is measured in minutes for this model. The ODEs are

$$\begin{aligned}
\frac{dG}{dt} &= -p_1 G - X(G + G_B) + u_g(t) \\
\frac{dX}{dt} &= -p_2 X + p_3 I \\
\frac{dI}{dt} &= -n(I + I_b) + \frac{1}{V_I} u_i(t)\,.
\end{aligned}$$

Typical parameter values are $p_1 = 0.01, p_2 = 0.025, p_3 = 1.3 \times 10^{-5}, V_I = 12, n = 0.093,\ G_B = 4.5, I_b = 15$.

The functions $u_g(t)$ and $u_i(t)$ model the infusion of glucose and insulin into the bloodstream. Specifically, $u_g(t)$ is the rate at which glucose is appearing, while u_i is the rate at which insulin is appearing.

The initial values are

$$G(0) = 0, X(0) = 0, I(0) = 0.05$$

(A) Draw a Simulink subsystem with two inputs: u_g, u_i for the meal glucose and meal insulin, respectively and one output $G(t)$.

(B) The control logic is a switched feedback controller with the following control law for the rate at which insulin is infused.

$$u_i(t) = \begin{cases} \frac{25}{3} & G(t) \leq 4 \\ \frac{25}{3}(G(t) - 3) & G(t) \in [4, 8] \\ \frac{125}{3} & G(t) \geq 8 \end{cases}$$

Model this in a control logic subsystem.

(C) The meal glucose model captures the rate at which the carbohydrates in a meal appear in the blood stream of the patient. A typical rate of appearance curve that is measured using trace-meal studies looks as follows:

Time Interval after meal (mins)	% of glucose appearing in interval
0 - 10	0 %
10 - 20	7 %
20 - 30	14 %
30 - 40	21 %
40 - 50	18 %
50 - 60	7 %
60 - 70	3 %
70 +	0 %

For instance, suppose a patient eats a meal with 110 gms of carbs at time T, then we can say that the value of $u_g(t)$ at time $T + 55$ is given by $\frac{7\% * 110}{10} = 0.77 gms/min$.

Given a meal specified by gms of carbs + time of meal (minute after simulation start), implement a meal glucose module that generates the value of $u_g(t)$ for that meal using the table above.

(D) Close the loop and simulate the closed loop system for different meal sizes at time $t = 20$. The meal sizes to be tried include $\{10gms, 20gms, 40gms, 80gms, 110gms, 125gms\}$. Simulate each scenario for $t \in [0, 240] mins$.

For each of the meal scenarios, compute the maximum and minimum values achieved for $G(t)$ from simulation, the glucose output.

See the file `igModel.slx` and attached script `runIGModel.m`. We obtain the following plot: