Minimax Option Pricing Meets Black-Scholes in the Limit

Objective

- Study an adversarial setting for pricing options
  - Zero-sum game: Nature vs. Investor
  - Option price is the value of the game
- Consider the limit as trade frequency increases
- Under reasonable constraints, what does Nature’s optimal strategy look like?
- How does our price compare to the commonly-used Black-Scholes price?

Options

- Types of option
  - European call/put - exercise at expiration
  - American call/put - exercise any time
  - “Exotic” -- basically any derivative
- Our setting
  - Exercised only at time $t = 1$
  - $X : [0, 1] \rightarrow \mathbb{R}$ is the price path
  - Payoff is $g(X(1))$ for $g$ convex
- European long call: $g(x) = \max(0, x - K)$

Replication Strategies

- Basic idea
  - A trading algorithm $A$ with initial debt $S_A$
  - Attempts to replicate the option $g$
  - Payoff dominates the option, minus $a$
- Key observation: $Price(g) \leq a$

The Black-Scholes Price

Price($g$) = $\mathbb{E}_{X \sim \text{GBM}}[g(X(1))]$

The Black-Scholes replication strategy

- Let $V(S, t)$ be the value of the option at $t$
- Let $\Delta \in A$ be the replication portfolio
- To solve for $V$, we need to solve a stochastic partial differential equation. Via Ito’s Lemma, we get
  \[ \frac{\partial V}{\partial S} - \frac{1}{2} \frac{\partial^2 V}{\partial S^2} = 0 \]
- Solution is:
  \[ V(S, t) = \mathbb{E}_{C \sim \text{GBM}}[g(SG(1-t))] \]

Black-Scholes, Kremer, Mansour, 2006

Option Pricing under Adversarial Assumptions

- What if prices were chosen by an adversary?
- Can we still replicate options in the worst case?
- DeMarzo et al: YES: Construct replication strategies via exponential-weight style algorithm
- Worst-case model gives a theoretical upper bound on the price of an option.

Our Game

- Nature vs. Investor

\[ \inf_{A} \sup_{X \in \mathcal{X}} \mathbb{E} \left[ g(X(1)) - \sum_{m} T_m \Delta_m \right] \]
- Discrete-time trades at $t = m/n, m \in [n]$
- $A \in \mathcal{A}$ is the replication algorithm (Investor)$A$ chooses $\Delta_m$ to invest at time $t = m/n$
- $X \in \mathcal{X}$ is the price path (Nature)
- $T_m$ is the fluctuation at time $t = m/n$
  \[ X(\frac{m}{n}) = X(\frac{m-1}{n}) (1 + T_m) \]
- Value of the game = option price!
- Interested in continuous trading limit as $n \to \infty$

Constraints on Nature

- Bounded per-round variance:
  \[ \mathbb{E} \left[ \left( \frac{X(t)}{X(s)} \right)^2 \right] \leq \exp(c(t-s)) - 1 \]
  - (GBM satisfies this with equality)
  - In particular, $\mathbb{E}[T_m^2 | T_{m-1}] \leq \exp(c/n) - 1$
- Bounded fluctuations:
  \[ |T_m| \leq \zeta_n \implies \zeta_n \to 0 \]
  - GBM does not satisfy this
  - For a lower bound, we must truncate GBM
  - Need $\zeta_n$ to increase relative to trading freq: $\lim_{n \to \infty} \frac{n \zeta_n}{\log n} > 16c$

The Proof

Step I: Duality

- Sion’s minimax theorem applies: $\inf \leftrightarrow \sup$
  \[ \sup_{X \in \mathcal{X}} \inf_{A \in \mathcal{A}} \mathbb{E} \left[ g(X(1)) - \sum_{m} T_m \Delta_m \right] \]
  - (By $\zeta_n$ fluctuation constraint, $\mathcal{X}$ is compact)

Step II: Martingality

- Claim: $T_1, \ldots, T_m$ is a martingale sequence
- Suppose not; then Investor can make some $T_m \Delta_m \to \infty$
- Value of the game is now $\sup_{X \in \mathcal{X}} \mathbb{E}[g(X(1))]$

Step III: Max Variance

- Since $g$ is convex, $T_m$ has maximal conditional variance for all $m$
- Hence, $\mathbb{E}[T_m^2 | T_{m-1}] = \exp(c/n) - 1$

Step IV: Finale

- Let $X^*_n$ be the optimal price path for $n$ trades
- Theorem: $X^*_n \to \text{GBM}$ as $n \to \infty$
- Corollary: $\lim_{n \to \infty} \mathbb{E}[g(X^*_n(1))] = \mathbb{E}[g(\text{GBM}(1))]$
- Option price approaches Black-Scholes price!

Conclusion

- The Black-Scholes pricing scheme is valid even in an adversarial model!
- Moreover, the stochastic assumption made by Black and Scholes can be derived as the optimal strategy of Nature in our model.

Future Work

- Allowing price jumps
  - Only consider points $X(1/n), \ldots, X(n/n)$
- Per-round variance $\to$ variance budget
  \[ \sum_{n=1}^{\infty} \mathbb{E}[T_n^2 | T_{n-1}] \leq c \]
- Theorem breaks: $X^*_n \not\to \text{GBM}$
- But final price converges: $X^*_n(1) \to \text{GBM}(1)$
- Hence we still have our Corollary:
  \[ \lim_{n \to \infty} \mathbb{E}[g(X^*_n(1))] = \mathbb{E}[g(\text{GBM}(1))] \]
- We still obtain the Black-Scholes price!