General Characterizations of Truthfulness via Convex Analysis

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November 29, 2012

Joint work with Ian Kash (MSRC)
Ian Says Hi!
Warm-up: Convex Functions

Definition

$G : \mathcal{T} \to \mathbb{R}$ is \textit{convex} if for all $x, y \in \mathcal{T}$ and all $\alpha \in [0, 1]$

$$\alpha G(x) + (1 - \alpha)G(y) \geq G(\alpha x + (1 - \alpha)y)$$
**Warm-up: Convex Functions**

**Definition**

A linear function $dG_t : \mathcal{T} \to \mathbb{R}$ is a *subgradient* to $G$ at $t$ if

$$\forall t' \in \mathcal{T} \quad G(t') \geq G(t) + dG_t(t' - t)$$
Fact

If $G_i$ are convex functions for $i \in I$, then $G$ is convex:

$$G(t) := \sup_{i \in I} G_i(t)$$
Mechanism Design

Single-player mechanism:

- Outcome space $\mathcal{O}$: possible allocations
- Type space $\mathcal{T} = (\mathcal{O} \to \mathbb{R})$: valuation functions
- Allocation rule $\alpha : \mathcal{T} \to \mathcal{O}$: reports to outcomes
- Payment rule $p : \mathcal{T} \to \mathbb{R}$: reports to payments

Bidder with type $t$ who reports $t' \in \mathcal{T}$ has net utility

$$U(t', t) = t(\alpha(t')) - p(t')$$

Truthfulness condition

$$\forall t, t' \in \mathcal{T} \quad U(t', t) \leq U(t, t)$$
Mechanism Design

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Mechanism Design

Single-player mechanism:

- Outcome space $\mathcal{O}$ possible allocations
- Type space $\mathcal{T} = (\mathcal{O} \to \mathbb{R})$ valuation functions
- Allocation rule $\alpha : \mathcal{T} \to \mathcal{O}$ reports to outcomes
- Payment rule $\rho : \mathcal{T} \to \mathbb{R}$ reports to payments

Bidder with type $t$ who reports $t' \in \mathcal{T}$ has net utility

$$U(t', t) = t(\alpha(t')) - \rho(t')$$

Truthfulness condition

$$\forall t, t' \in \mathcal{T} \quad U(t', t) \leq U(t, t)$$
For single-parameter mechanisms:

**Theorem**

\( \alpha \) is implementable \( \iff \) \( \alpha \) is monotone

*Implementable means payments \( p \) making \( (\alpha, p) \) truthful*

Equivalently:

**Theorem**

\( \alpha \) is implementable \( \iff \exists G : \mathcal{T} \to \mathbb{R} \) convex s.t. \( \alpha \) is a subgradient to \( G \)

*\( G \) is the consumer surplus*
For single-parameter mechanisms:

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\[ \alpha \text{ is implementable} \iff \exists G : \mathcal{T} \rightarrow \mathbb{R} \text{ convex s.t. } \alpha \text{ is a subgradient to } G \]

*G is the consumer surplus*
Scoring Rules

- **Outcome space** $\mathcal{O}$  
  mutually exclusive events
- **Private belief** $p \in \Delta_\mathcal{O}$  
  probabilities over outcomes
- **Scoring rule** $S : \Delta_\mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$  
  score of report given an outcome

Expected score of report $p'$ given truth $p$ is

$$S(p', p) := \mathbb{E}_{o \sim p} [S(p', o)]$$

**Truthfulness condition**

$$\forall p, p' \quad S(p', p) \leq S(p, p)$$
Scoring Rules

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### Truthfulness condition

$$\forall p, p' \quad S(p', p) \leq S(p, p)$$
Gneiting and Raftery 2007

Theorem

Scoring rule $S$ is truthful $\iff$ there is some convex $G : \Delta_1 \rightarrow \mathbb{R}$ with subgradients $\{dG_p\}$ such that

\[ S(p, o) = G(p) + dG_p (1_o - p) \]
What’s the Connection?

Mechanism:
- Outcomes $\mathcal{O}$
- Type $\mathcal{T} = (\mathcal{O} \rightarrow \mathbb{R})$
- Utility $U(t', t)$

Scoring rule:
- Outcomes $\mathcal{O}$
- Belief $p \in \Delta_{\mathcal{O}}$
- Score $S(p', p)$

Truthfulness

$U(t', t) \leq U(t, t)$

$U(t', t) = t(\alpha(t')) - p(t')$

$= (t, 1_{\alpha}(t)) - p(t')$

Truthfulness

$S(p', p) \leq S(p, p)$

$S(p', p) := \mathbb{E}_{o \sim p} [S(p', o)]$

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Reward affine in private info!
What’s the Connection?

Mechanism:
- Outcomes $O$
- Type $T = (O \to \mathbb{R})$
- Utility $U(t', t)$

Scoring rule:
- Outcomes $O$
- Belief $p \in \Delta_O$
- Score $S(p', p)$

Truthfulness

$U(t', t) \leq U(t, t)$

$S(p', p) \leq S(p, p)$

$U(t', t) = t(\alpha(t')) - p(t')$

$= (t, \mathbb{1}_a(p)) - p(t')$

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$= (t, 1_{\alpha(t')}) - p(t')$

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#### Truthfulness

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$U(t', t) \leq U(t, t)$

\[
U(t', t) = t(a(t')) - p(t')
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= \langle t, 1_{a(t')} \rangle - p(t')
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Reward *affine* in private info!
What’s the Connection?

Mechanism:
- Outcomes $\mathcal{O}$
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- Utility $U(t', t)$

Truthfulness

$$U(t', t) \leq U(t, t)$$

$$U(t', t) = t(a(t')) - p(t')$$
$$= \left\langle t, 1_{a(t')} \right\rangle - p(t')$$

Scoring rule:
- Outcomes $\mathcal{O}$
- Belief $p \in \Delta_{\mathcal{O}}$
- Score $S(p', p)$

Truthfulness

$$S(p', p) \leq S(p, p)$$

$$S(p', p) := \mathbb{E}_{o \sim p} \left[ S(p', o) \right]$$
$$= \left\langle p, S(p', \cdot) \right\rangle$$

Reward *affine* in private info!
Our Model: Affine Score

- Type space $\mathcal{T}$ \textit{any subset of a vector space}
- Reward space $\mathcal{A} \subseteq \text{Aff}(\mathcal{T} \rightarrow \mathbb{R})$ \textit{affine functions on types}
- Affine score $S : \mathcal{T} \rightarrow \mathcal{A}$

Truthfulness condition

$S(t')(t) \leq S(t)(t)$

Observation: $G(t) := \sup_{t'} S(t')(t)$ convex

and $S$ truthful $\implies G(t) = S(t)(t)$
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A General Truthfulness Characterization

**Theorem**

Affine score $S : \mathcal{T} \rightarrow \mathcal{A}$ is truthful if and only if there exists some convex $G : \text{Conv}(\mathcal{T}) \rightarrow \mathbb{R}$, and subgradients $\{dG_t\}$, such that

$$S(t')(t) = G(t') + dG_{t'}(t - t').$$

- Techniques from Gneiting-Raftery and Archer-Kleinberg
- Immediately gives previous scoring rule and mechanism characterizations
Proof: Convex $\mathcal{T}$

Proof of $\Leftarrow$:

- $S(t')(t) = G(t') + dG_{t'}(t - t')$
  \[ \leq G(t) = S(t, t) \quad \text{by def. subgradient} \]

Proof of $\Rightarrow$:

- $G(t) := \sup_{t'} S(t')(t)$  \quad \text{convex as pointwise supremum!}
- Define $dG_{t'}(\cdot) = S_{t'}(\cdot)$ \quad \text{linear part of } S(t)
- $S(t)(\cdot)$ subgradient to $G$ at $t$: \quad \text{by truthfulness}

\[ G(t') + dG_{t'}(t - t') = S(t')(t) \leq G(t) \]
Proof: Convex $T$

Proof of $\iff$:

- $S(t')(t) = G(t') + dG(t')(t - t')$
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Proof of $\implies$:

- $G(t) := \sup_{t'} S(t')(t) \quad \text{convex as pointwise supremum!}$
- Define $dG(t)(\cdot) = S_{t}(t)(\cdot) \quad \text{linear part of } S(t)$
- $S(t)(\cdot)$ subgradient to $G$ at $t$: \quad \text{by truthfulness}
  \[
  G(t') + dG(t')(t - t') = S(t')(t) \leq G(t)
  \]
Proof: Convex $\mathcal{T}$

Proof of $\leftarrow$:
- $S(t')(t) = G(t') + dG_t'(t - t')$
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Proof of $\rightarrow$:
- $G(t) := \sup_{t'} S(t')(t)$ \quad \text{convex as pointwise supremum!}
- Define $dG_t(\cdot) = S_\ell(t)(\cdot)$ \quad \text{linear part of } S(t)
- $S(t)(\cdot)$ subgradient to $G$ at $t$: \quad \text{by truthfulness}

$$G(t') + dG_t'(t - t') = S(t')(t) \leq G(t)$$
Proof: Convex $\mathcal{T}$

Proof of $\Leftarrow$:

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Proof: Non-Convex $\mathcal{T}$

Proof of $\Leftarrow$: same.

Proof of $\Rightarrow$:

- Consider $\hat{t} \in \text{Conv}(T) \setminus T$
- Write $\hat{t} = \sum_i \alpha_i t_i$ for $t_i \in T$
- Define $S(t)(\hat{t}) = \sum_i \alpha_i S(t)(t_i)$
- Define $G(\hat{t}) = \sup_{t \in T} S(t)(\hat{t})$

Proceed as before...
Proof: Non-Convex $\mathcal{T}$

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Proceed as before...
Immediate New Results

1. Proper scoring rules for non-convex sets of distributions
   Fewer constraints $\implies$ more scoring rules?

2. “Local” mechanisms and scoring rules
   Convexity is a local property
Immediate New Results

1. Proper scoring rules for non-convex sets of distributions
   * Fewer constraints $\implies$ more scoring rules?

2. “Local” mechanisms and scoring rules
   * Convexity is a local property
Mechanism Design: Implementability of $\alpha$

**Definition**

$\{dG_t\}_{t \in \mathcal{T}}$ satisfies **cyclic monotonicity (CMON)** if for all finite sets $\{t_0, \ldots, t_k\} \subseteq \mathcal{T}$,

$$\sum_{i=0}^{k} dG_{t_i}(t_{i+1} - t_i) \leq 0.$$

CMON with $k = 2$ is **Weak monotonicity (WMON)**.

Let $L_{xy} = \int_{0}^{1} dG_{\beta y + (1-\beta)x} (y - x) \, d\beta$.

**Definition**

$\{dG_t\}_{t \in \mathcal{T}}$ satisfies **path independence (PI)** if for all $x, y, z \in \mathcal{T}$

$$L_{xy} + L_{yz} = L_{xz}$$
Previous Characterizations

CMON

- Implementable
  - Rochet 1987
  - Myerson 1981

- Subgradient
  - Archer, Kleinberg 2008

- Müller et al. 2007
  - WMON + PI
  - LWMON + VF
A New Proof Structure

Implementable

Thm 1

WMON + PI

Thm 3

Subgradient

CMON

Thm 2

Cor 6

LWMON + VF

Thm 4

WL Subgradient
Reproving Müller et al.

New proof via construction of $G$:

- Fix $G(t_0)$
- Extend $G(t) = L_{t_0 t}$ integrable by WMON, consistent by PI
- Subgradient by simple computation
Reproving Müller et al.

New proof via construction of $G$:

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**Q:** What if types are exponential (or infinite!) in size?

**A:** Use summary information / low-dim representation

Examples:
- Scoring rules for statistics  [Lambert-Pennock-Shoham, Gneiting]
- Rankings instead of utilities  [Carroll]
- ...

Q: What if types are exponential (or infinite!) in size?
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More Formally...

Wish to change report space from $\mathcal{T}$ to some other $R$

$$S : R \rightarrow \text{Aff}(\mathcal{T} \rightarrow \mathbb{R}); \quad S(r)(t)$$

What does truthful mean now?
Definition

A *property* is a map $\Gamma : \mathcal{T} \rightarrow R$ specifying the correct report $r = \Gamma(t)$ for each type $t$.

Truthfulness condition

$$S(r')(t) \leq S(\Gamma(t))(t)$$

We say $S$ *elicits* $\Gamma$. 
Definition

A property is a map $\Gamma : \mathcal{T} \rightarrow R$ specifying the correct report $r = \Gamma(t)$ for each type $t$.

Truthfulness condition

$$S(r')(t) \leq S(\Gamma(t))(t)$$

We say $S$ elicits $\Gamma$. 
A New Result

Theorem

Property $\Gamma$ is elicitable iff there exists $G : \mathcal{T} \rightarrow \mathbb{R}$ differentiable and convex, and map $\varphi : \mathbb{R} \rightarrow \nabla G(T)$, such that $\varphi(\Gamma(t)) = \nabla G(t)$.

New insights:

- Elicitable properties $==$ subgradients!
- Properties specify where $G$ should be flat
Theorem

Property $\Gamma$ is elicitable iff there exists $G : \mathcal{T} \rightarrow \mathbb{R}$ differentiable and convex, and map $\phi : \mathbb{R} \rightarrow \nabla G(\mathcal{T})$, such that $\phi(\Gamma(t)) = \nabla G(t)$.

New insights:

- Elicitable properties $\equiv$ subgradients!
- Properties specify where $G$ should be *flat*
Finite $R$: Power Diagram

Cells = types with same report. Application: rankings!
Thanks!