Minimax Option Pricing: How Robust is Black-Scholes?

Rafael Frongillo

Department of Computer Science
University of California at Berkeley

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Joint work with Jake Abernethy and Andre Wibisono
Jake Abernethy
now at UPenn

Andre Wibisono
in 1225 this summer!
A new financial instrument which is a function of old ones.

Class of derivatives we consider:

- Expiration date $T$ (typically 1)
- Base stock/asset $S$
- Derivative pays out $g(S(T))$ at time $T$

$S(t)$ is the value of $S$ at time $t$

E.g. $\cos(\text{gas price on Aug 1})$
Financial Derivatives

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Running example: European call option

\[ g(S) = \max(0, S - K), \text{ where } K \text{ is the strike price} \]

Note: will use “option” and “derivative” interchangeably
How to Price?

What is a derivative $g$ worth?

![Graph showing option price versus stock price](graph.png)

- $A$: Option price curve
- $B$: Option price curve
- $T_1$, $T_2$, $T_3$: Time periods

(Exercise Price = $20)
Fischer Black and Myron Scholes, 1973

- Intuition: price of derivative is cost of implementing it with existing instruments
- The algorithm which implements a derivative is a *replication strategy*
- The replication strategy has a fixed initial investment, which should be precisely the price of the derivative
Replication Strategies

Idea: As stock $S$ fluctuates, use an algorithm $A$ to “hedge” the option by buying and selling $S$

Result: guarantee the payoff of the option, minus a fixed cost $c$
Black-Scholes Assumptions

- No arbitrage opportunities
- 0% interest borrowing
- Can trade continuously
- No transaction fees, no dividend payments, etc
- Stock prices follow Geometric Brownian Motion (GBM)
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Geometric Brownian Motion
Let $W(t)$ be Brownian Motion with drift $\mu$ and volatility $\sigma^2$

- $W(0) = 0$
- $W(t) - W(s)$ and $W(u) - W(t)$ are indep. for $s < t < u$
- $W(t) - W(s) \sim N(\mu(t - s), \sigma^2(t - s))$

$G(t)$ is GBM $\iff \log(G(t))$ is Brownian Motion
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Delta-hedge Portfolio

Given option/derivative $g$:

- Let $V(S, t)$ be the value of the option at $t$
- Let $\frac{\partial V}{\partial S}$ be the replication portfolio

$\text{Hold } \$\frac{\partial V}{\partial S}(t) \text{ of stock @ time } t$

Now solve for $V$ using the no-arbitrage condition:

- Stochastic PDE from Ito's Lemma:

$$\frac{\partial V}{\partial S} - \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

- Solution is:

$$V(S, t) = \mathbb{E}_{G \sim \text{GBM}}[g(SG(T - t))]$$
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The Black-Scholes Price

Price of option is therefore:

\[ V(S, 0) = \mathbb{E}[g(S \text{ GBM}(T))] \]

Some surprises:
- Replication succeeds with probability 1!
- GBM above has drift 0 not \( \mu \)!
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Beyond Black-Scholes

Problems with Black-Scholes

- Continuous-time trading
- Assumes GBM!

Why stochastic prices?

*Prices respond to decisions of other traders!*

Why not *adversarial* prices? [DeMarzo, Kremer, Mansour ’08]
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An Option-Pricing Game

\[ \inf_{A \in A} \sup_{X \in X} \mathbb{E} \left[ g(X(1)) - \sum_{m=1}^{n} T_m \Delta_m \right] \]

- An \( n \)-round game between Investor and Nature
- Discrete-time trades at \( t = m/n, m \in [n] \)
An Option-Pricing Game

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- \( A \in \mathcal{A} \) is the replication algorithm (Investor)
- \( A \) chooses \( \Delta_m \) to invest in \( S \) at time \( t = m/n \)
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- \( X \in \mathcal{X} \) is the price path (Nature)
- \( T_m \) is the fluctuation at time \( t = m/n \):
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  X \left( \frac{m}{n} \right) = X \left( \frac{m-1}{n} \right) (1 + T_m)
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- Earnings of Investor
- Difference = “Regret”
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Value of the game \( \geq \) option price!

*Upper bound because of the worst-case assumptions*

Interested continuous trading limit as \( n \to \infty \)
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Constraining Nature

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\inf_{A \in A} \sup_{X \in \mathcal{X}} \mathbb{E} \left[ g(X(1)) - \sum_{m=1}^{n} T_m \Delta_m \right]
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What price paths \( \mathcal{X} \) can Nature choose from?

We require:

\[
\mathbb{E}[T_m^2 | T_{m-1}] \leq \frac{c}{n}
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\( c \) is the "volatility"
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By Sion’s Minimax Theorem, we can swap inf and sup!
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Now \( \{T_m\} \) must be a martingale sequence

- Assume not: \( \mathbb{E}[T_m|T_{m-1}] \neq 0 \)
- Investor can choose \( \Delta_m \to \pm \infty \)
- Nature would have unbounded loss!

But now the algorithm is completely irrelevant!
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When \( g \) is convex, Nature wants to maximize variance

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Similar reasoning to the Maximum Principle.
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Step IV: Central Limit Theorem

Let $X^*_n$ be Natures OPT price path at $n$

*Martingale sequence with conditional variance $c/n$

Applying a martingale CLT: Lindeberg–Feller Theorem

**Theorem**

As $n \to \infty$, $X^*_n \xrightarrow{d} GBM$

**Corollary**

As $n \to \infty$, $\mathbb{E}[g(X^*_n(1))] \to \mathbb{E}[g(GBM(1))]$
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\begin{itemize}
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  \item \textbf{Corollary}
  \begin{align*}
    \text{As } n \to \infty, \quad \mathbb{E} \left[ g(X_n^*(1)) \right] & \to \mathbb{E} \left[ g(GBM(1)) \right]
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\end{itemize}

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Value of the game Black-Scholes price!
What Just Happened?

## Black-Scholes Option Pricing

- Assume stock $\sim$ GBM
- Construct optimal replication strategy

$$\text{Price}(g) = \mathbb{E}[g(\text{GBM}(1))]$$

## Minimax Option Pricing

- Assume stock is adversarial
- Analyze *dual* of the game
- Worst-case price path $\rightarrow$ GBM

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$$\text{Price}(g) = \mathbb{E}[g(\text{GBM}(1))]$$
Our constraint on Nature:

$$\mathbb{E}[T_m^2 | T_{m-1}] \leq \frac{c}{n}$$

[DeMarzo, Kremer, Mansour '08] use a *cumulative* constraint:

$$\sum_{m=1}^{n} \mathbb{E}[T_m^2 | T_{m-1}] \leq c$$

- Weaker constraint
- Allows for price jumps

*GBM is continuous w.p. 1*
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*GBM is continuous w.p. 1*
Still Obtain Black-Scholes Price?

From [DeMarzo, Kremer, Mansour ’08]:

![Graph showing the comparison of Black-Scholes, optimal bound, and learning algorithm for option value against strike price.](image-url)
Some Speculation

We believe:

- \( X_n^* \rightarrow \text{GBM} \)
- \( X_n^*(1) \rightarrow \text{GBM}(1) \)

Hence, we would still obtain the Black-Scholes price!

Proof ideas:

- \( \text{support}(T_m) = 2 \) in dual game
- Optimal \( \Delta_m \) balances these two points
- Then \( \Delta_m \) is a discrete derivative of \( V \)
- This \( V \) approaches Black-Scholes \( V \), and \( \Delta_m \) approaches the delta-hedge portfolio!
Some Speculation

We believe:

- $X_n^* \not\rightarrow \text{GBM}$
- $X_n^*(1) \rightarrow \text{GBM}(1)$

Hence, we would still obtain the Black-Scholes price!

Proof ideas:

- $\text{support}(T_m) = 2$ in dual game
- Optimal $\Delta_m$ balances these two points
- Then $\Delta_m$ is a discrete derivative of $V$
- This $V$ approaches Black-Scholes $V$, and $\Delta_m$ approaches the delta-hedge portfolio!
Consider the value function for this game:

\[
V_n(S, n) := g(S)
\]
\[
V_n(S, m) := \inf_\Delta \sup_{t \in [-z, z]} \Delta t + V_n(S(1 + t), m - 1)
\]

And let \( \Delta = \Delta(S, m) \) be the optimal investment for Investor

**Lemma**

*If \( \Delta = \Delta(S, m) \), then Nature's sup \( t \) is achieved by at least two points \( t_1, -t_2 \) with \( t_1, t_2 > 0 \)*
Consider the value function for this game:

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If \( \Delta = \Delta(S, m) \), then Nature's \( \sup_t \) is achieved by at least two points \( t_1, -t_2 \) with \( t_1, t_2 > 0 \).
By the Lemma, $\Delta$ must balance $V_n(S, m - 1)$ at $t_1$ and $-t_2$:

$$V_n(S, m) = \Delta(S, m) t_1 + V_n(S(1 + t_1), m - 1)$$

$$= -\Delta(S, m) t_2 + V_n(S(1 - t_2), m - 1)$$

Solving for $\Delta$:

$$\Delta(S, m) = \frac{V_n(S(1 - t_2), m - 1) - V_n(S(1 + t_1), m - 1)}{t_1 + t_2}$$

Foreshadowing

A discrete derivative... reminiscent of the delta-hedge portfolio!
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Martingale??

Plugging $\Delta$ back in:

$$V_n(S, m) = \frac{t_1}{t_1 + t_2} V_n(S(1-t_2), m-1) + \frac{t_2}{t_1 + t_2} V_n(S(1+t_1), m-1)$$

Introduce a random variable $T = \begin{cases} t_1 & \text{w.p. } \frac{t_2}{t_1 + t_2} \\ -t_2 & \text{w.p. } \frac{t_1}{t_1 + t_2} \end{cases}$

Note $\mathbb{E}[T] = 0$

$$V_n(S, m) = \mathbb{E}_T \left[ V_n(S(1 + T), m - 1) \right]$$
Plugging $\Delta$ back in:

$$V_n(S, m) = \frac{t_1}{t_1 + t_2} V_n(S(1-t_2), m-1) + \frac{t_2}{t_1 + t_2} V_n(S(1+t_1), m-1)$$

Introduce a random variable $T = \begin{cases} t_1 & \text{w.p. } \frac{t_2}{t_1 + t_2} \\ -t_2 & \text{w.p. } \frac{t_1}{t_1 + t_2} \end{cases}$

Note $\mathbb{E}[T] = 0$

$$V_n(S, m) = \mathbb{E}_T \left[ V_n(S(1+T), m-1) \right]$$
Martingale??

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$$V_n(S, m) = \mathbb{E}_T \left[ V_n(S(1 + T), m - 1) \right]$$
Applying this at every round:

$$V_n(S, 0) = \mathbb{E} \left[ V_n \left( S \cdot \prod_{m=1}^{n} (1 + T_m), n \right) \right]$$

= \mathbb{E} \left[ g \left( S \cdot \prod_{m=1}^{n} (1 + T_m) \right) \right]

Conjectures

1. $V_n(S, n) \longrightarrow V_{B-S}(S, 1)$
2. $\Delta(S, m) \longrightarrow \frac{\partial}{\partial S} V_{B-S}(S, \frac{m}{n})$
thank you