A SOLUTION CONCEPT FOR GAMES WITH ALTRUISM AND COOPERATION

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ABSTRACT. We propose a new solution concept for one-shot normal form games based on a principle that can be summarized as follows: players forecast how the game would be played if they formed coalitions and then they play according to their best forecast.

We prove that this cooperative equilibrium exists for all finite games and it meets the experimental data collected for the Prisoner’s dilemma, the Traveler’s dilemma, Nash bargaining problem, Bertrand competition, public good game, dictator game, ultimatum game, Hawk-Dove, and other specific games for which other solution concepts fail to predict human play.

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1. Introduction

Since the foundation by Morgenstern and von Neumann [Mo-vN44], the major challenge of modern game theory has been to predict which actions a human player would adopt in a strategic situation. A first prediction was proposed in an earlier paper by J. von Neumann [vN28] for two-person zero-sum games and then generalized to every finite game by J. Nash in [Na50a]. Since then Nash equilibrium has been certainly the most notable and used solution concept in game theory. Nevertheless, over the last sixty years, it has been realized that Nash equilibrium makes poor predictions of human play and, indeed, a large number of experiments have been conducted on games for which it dramatically fails to predict human behavior.

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There are many reasons behind this failure. On one hand, when there are multiple equilibria, it is not clear which one we should expect is going to be played. A whole stream of literature, finalized to the selection of one equilibrium, arose from this point, including the definitions of evolutionary stable strategy [MS-P73], perfect equilibrium [Se75], trembling hand perfect equilibrium [Se75], proper equilibrium [My78], sequential equilibrium [Kr-Wi82], limit logit equilibrium [MK-Pa95], and, very recently, settled equilibrium [My-We12].

On the other hand, the criticism of Nash equilibrium is motivated by more serious problems: there are examples of games with a unique Nash equilibrium which is not played by human players. Typical examples of such a fastidious situation are the Prisoner’s Dilemma [Fl52] and the Traveler’s Dilemma [Ba94]. This point has motivated another stream of literature devoted to the explanation of such deviations from Nash equilibria. Part of this literature tries to explain such deviations assuming that players make mistakes in the computation of the expected value of a strategy and therefore, assuming that errors are identically distributed, a player may play also non-optimal strategies with a probability described by a Weibull distribution. This intuition led to the foundation of the so-called quantal response equilibrium theory by McKelvey and Palfrey [MK-Pa95]. A variant of this theory, called quantal level-k theory and proposed by Stahl and P. Wilson in [St-W94], was recently shown to perform better in the prediction of human behavior [Wr-LB10]. In the same paper, Wright and Leyton-Brown have shown also that quantal level-k theory predicts human behavior significantly better than all other behavioral models that have been proposed in the last decade, as the level-k theory [CG-Cr-Br01] and the cognitive hierarchy model [Ca-Ho-Ch04]. However, a first criticism of quantal level-k theory is that it is not scale invariant, contradicting one of the axioms of expected utility theory of Morgenstern and von Neumann [Mo-vN47]. A major criticism stems from the fact that this theory only makes use of some parameters describing either the incidence of errors that a player can make computing the expected utility of a strategy or the fact that humans can perform only a bounded number of iterations of strategic reasoning. This has two negative counter-parts: first this theory is not predictive, in the sense that one has to conduct experiments to estimate the parameters; second, this theory intrinsically affirms that deviation from Nash equilibria can descend only from two causes, computational mistakes and bounded rationality, that are hard to justify for games with very easy payoffs, like the Prisoner’s Dilemma, or for games where the deviation from Nash equilibrium is particularly strong, like the Traveler’s Dilemma with small bonus-penalty. For instance, the unique Nash equilibrium for the Traveler’s dilemma with bonus/penalty $b = 2$ and strategy sets \( S_1 = S_2 = \{2, 3, \ldots, 100\} \) it \((2, 2)\), but it has been reported that most of the players chose strategy between 96 and 100 (cf. Example 5.1). This enormous difference cannot be explained using only bounded rationality or computational mistakes and the general feeling is that something deeper is going on. Indeed, there is an increasing belief that Nash equilibrium should be replaced by a conceptually different solution concept that coincides with Nash equilibrium only in particular cases. The first studies in this direction have been presented by Halpern and Rong [Ha-Ro10], by Halpern and Pass [Ha-Pa11], by Renou and Schlag [Re-Sc09] and Halpern and Pass [Ha-Pa12], and by Adam and Ehud Kalai [Ka-Ka13]. Nevertheless, even though these solution concepts can explain deviations from Nash equilibria in some games, all of them make unreasonable predictions for at least one game of interest. For instance, the maximum perfect cooperative equilibrium introduced in [Ha-Ro10] is too rigid and predicts cooperation for sure in the
Prisoner’s and Traveler’s Dilemmas, contradicting the experimental data collected in [Co-DJ-Fo-Ro96], [Ca-Go-Go-Ho99], [Go-Ho01], [Be-Ca-Na05], and [Ba-Be-St11]. The *iterated regret minimization* procedure introduced in [Re-Sc09] and [Ha-Pa12] can explain deviations towards cooperation in some variants of the Traveler’s Dilemma\(^1\), the Bertrand competition, the Centipede Game, and other games of interest, but it does not predict deviation towards cooperation in finite iterations of the Prisoner’s Dilemma\(^2\), it cannot explain altruistic behaviors in the ultimatum game and in the dictator game, and makes unreasonable predictions for the Traveler’s dilemma with punishment (see Example 5.9), and a certain zero-sum game (see Example 8.3). The solution concept defined using algorithmic rationality in [Ha-Pa11] can explain deviation towards cooperation in the iterated Prisoner’s and Traveler’s dilemmas, but it does not predict deviation towards cooperation in one-shot versions of the Prisoner’s dilemma or in one-shot versions of the Traveler’s dilemma with very small bonus-penalty, contradicting the experimental data reported in [Co-DJ-Fo-Ro96] and [Ca-Go-Go-Ho99]. Finally, the coco value introduced by Adam and Ehud Kalai in [Ka-Ka13], unifying and developing previous works by Nash [Na53], Raiffa [Rai53], and E.Kalai-Rosenthal [Ka-Ro78], also appears to be too rigid. For instance, if two agents played the Prisoner’s dilemma according to the coco value, then they would both cooperate for sure. This prediction contradicts the experimental data collected in [Co-DJ-Fo-Ro96].

In this paper we try to reconduct the failure of all these attempts to two basic problems.

On one hand, the use of utility functions in the very definition of a game is problematic for two reasons. The first one is the experimental evidence that expected utility theory fails to predict the behavior of decision makers [Al53], [Ka-Tv00], [St00]. The second one is more conceptual: since the translation of a gain function (that is, what we really know about a game) to a utility function (that is, what the classical theory uses to make predictions) is an highly non-trivial problem to solve without falling into circular definitions, it seems that a theory based on utility functions risks seriously not to satisfy Popper’s falsifiability principle [Po59].

This problem could be theoretically overcome replacing utility functions with gain functions and applying Kahneman-Tversky’s cumulative prospect theory [Tv-Ka92]. But one can easily convince himself that in most cases such a replacement could explain only *quantitative deviations*.

The second problem is indeed that experiments conducted on the Prisoner’s dilemma, the Traveler’s dilemma, and other games, show *qualitative deviations* from classical solution concepts. These qualitative deviations suggest that humans have attitude to cooperation.

These observations motivate the definition of a new solution concept, able to take into account cooperation and using gain functions instead of utility functions. This paper represents a first endeavour in this direction. Indeed, here we consider only one-shot simultaneous-move normal form games. The aim of this paper is to define a solution

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1Indeed, it predicts a too quick convergence to Nash equilibrium, as the bonus/penalty increases (see Example 5.1).

2Even though the authors can actually overcome this problem allowing prior beliefs (cf. [Ha-Pa12], Section 3.5). Nevertheless, prior beliefs of the kind allowed in [Ha-Pa12] cannot explain deviation towards cooperation that has been observed even in one-shot versions of the Prisoner’s dilemma [Co-DJ-Fo-Ro96].

3We call *gain function* a function whose values are the monetary outcomes or, more generally, the quantity of some good that is won or lost by a player.
concept, called cooperative equilibrium, for this class of games formalizing the following principle of cooperation:

(C) Players try to forecast how the game would be played if they formed coalitions and then they play according to their best forecast.

The study of games assuming existence of cooperative people is not a new idea (cf. [Fe-Sc99]), but, to the best of our knowledge, this is the first attempt to lift up this well known tendency to cooperation to a general principle which is nothing else that a deeper and smarter realization of selfishness.

The idea to formalize the principle of cooperation and define the cooperative equilibrium can be briefly summarized as follows:

- We assume that players do not act a priori as single players, but they try to forecast how the game would be played if they formed coalitions.
- These forecasts are representend by a number $v_i(p)$, called value of the coalition $p$ for player $i$, which is a measure of the expected gain of player $i$ when she plays according to the coalition $p$.
- The numbers $v_i(p)$ induce a sort of common beliefs: we consider the induced game $\text{Ind}(G, p)$ which differs from the original game $G$ only for the sets of allowed profiles of mixed strategies: the profiles of mixed strategies allowed in $\text{Ind}(G, p)$ are the profiles $(\sigma_1, \ldots, \sigma_N)$ such that $u_i(\sigma_1, \ldots, \sigma_N) \geq v_i(p)$, for any player $i$.
- The exact cooperative equilibrium is one where player $i$ plays an equilibrium of the game $\text{Ind}(G, p)$ induced by a coalition which maximizes the value function $v_i$.
- The notion of equilibrium for the induced game $\text{Ind}(G, p)$ is not defined using classical Nash equilibrium, but using a prospect theoretical analogue.

In order to apply prospect theory and in order to construct a falsifiable theory, we must replace utility functions by gain functions, that are, functions whose values represent the monetary outcomes or, more generally, the quantity of some good which is won or lost by a player. This replacement comes at the price that we must take into account explicitly new data that were implicitly included in the utility functions. These data are the fairness functions $f_i$ and the altruism parameters $a_{ij}$. We are not the first researchers to make explicit some parameters in order to explain deviations from standard theories. For instance, Fehr and Schmidt [Fe-Sc99] used parameters $\alpha_i, \beta_i$ (two parameters for each player) to explain deviations from Nash equilibrium in the ultimatum game, the public good game with punishment, and other games. Other behavioral models do not introduce explicitly new parameters, but assume that the population is divided in different types of subjects (e.g., altruistic and selfish subjects). These additional parameters and these additional assumptions make the respective theories descriptive, in the sense that they cannot predict the result of an experiment, but only explain it. We also use new parameters, but we will show that, in many cases, they do not play any active role: they theoretically exist, but the cooperative equilibrium does not depend on them. This implies that the cooperative equilibrium is a predictive solution concept for many games of interest. A bit more precisely, in this paper we prove the following statements.

\[4\] The word exact means that, since players can have bounded rationality or can make mistakes in the computations, one can also define a quantal cooperative equilibrium borrowing ideas from quantal response equilibrium and quantal level-$k$ theory and say that player $i$ plays with probability $e^{\lambda v_i(p)} / \sum_p e^{\lambda v_i(p)}$ a quantal response equilibrium or a quantal level-$k$ equilibrium of the game $\text{Ind}(G, p)$.\]
Fact 1.1. The cooperative equilibrium for the Prisoner’s dilemma is predictive (i.e., it does not depend on fairness functions and altruism parameters) and it fits the experimental data.

Fact 1.2. The cooperative equilibrium for the Traveler’s dilemma is predictive and it fits the experimental data.

Fact 1.3. The cooperative equilibrium for the Bertrand competition is predictive and it fits the experimental data.

Fact 1.4. The cooperative equilibrium for the public good game is predictive and it fits the experimental data.

Fact 1.5. The cooperative equilibrium fits Kahneman-Knetsch-Thaler’s experiment related to the ultimatum game.

More generally, we will describe a large class of games for which the cooperative equilibrium is predictive. The description of this class requires some definitions and therefore it is postponed (see Corollary 9.7).

We also prove

Fact 1.6. The cooperative equilibrium predicts the (50,50) solution in the Bargaining problem under natural assumptions on the fairness functions.

Fact 1.7. The cooperative equilibrium explains the experimental data collected for the dictator game, via altruism.

Another case where the cooperative equilibrium is only descriptive is when the mistakes that players can make in the computations have a very strong influence on the result. A typical example

Fact 1.8. The quantal cooperative equilibrium explains Goeree-Holt’s experiment about the asymmetric matching pennies.

The structure of the paper is as follows. In Section 2, we define the so-called games in explicit form (see Definition 2.2), where the word explicit really means that we have to take into account explicitly new data (altruism parameters and fairness functions). In Section 3 we describe informally the idea through a simple example that allows to motivate all main definitions of the theory. In Section 4 we define the cooperative equilibrium for games in explicit form under expected utility theory, that is, without using cumulative prospect theory, and without using the altruism parameters (see Definition 4.12). The reason of this choice is that in most cases cumulative prospect theory can change predictions only quantitatively and not qualitatively and that, in most cases, altruism parameters do not play any active role. Indeed, we compute the cooperative equilibrium (under expected utility theory and without using the altruism parameters) for the Prisoner’s Dilemma (see Examples 4.5, 5.8, and 6.1), Traveler’s Dilemma (see Examples 4.6 and 5.1), Nash bargaining problem (see Example 4.4 and 5.6), Bertrand competition (see Example 5.2), Hawk-Dove (see Example 5.10), public good game (see Example 5.5), the ultimatum game (see Example 5.3), and a specific game of particular interest since iterated regret minimization theory fails to predict human behavior,\footnote{Roughly speaking, the assumption is that the two players have the same perception of money. We believe that this assumption is natural, since it is predictable that a bargain between a very rich person and a very poor person can have different solution.}
whereas the cooperative equilibrium does (see Example 3.9). We make a comparison
between the predictions of the cooperative equilibrium and the experimental data
and we show that they are always close. In Section 6 we discuss a few examples
where the replacement of expected utility theory by cumulative prospect theory
starts playing
an active role (see Examples 6.1, 6.2, and 6.3). Here it starts the ideal second part of
the paper, devoted to the definition of the cooperative equilibrium for games in ex-
plicit form, using cumulative prospect theory and taking into account altruism. Before
doing that, we take a short section, namely Section 7, to give a brief introduction
to cumulative prospect theory. The definition of the cooperative equilibrium under
cumulative prospect theory and taking into account altruism takes Sections 8 and 9: in
the former we define a procedure of iterated deletion of strategies using the altruism
parameters and we apply it to explain the experimental data collected for the dictator
game (see Example 8.10); in the latter we repeat the construction done in Section 4,
this time under cumulative prospect theory instead of expected utility theory. Theorem
9.6 shows that all finite games have a cooperative equilibrium. Part of Section 8 may
be of intrinsic interest, since it contains the definition of super-dominated strategies
(see Definition 8.1) and their application to solve a problem left open in \cite{Ha-Pa12}
(see Example 8.12). Section 10 states a few important problems that should be addressed
in future researches.

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2. Utility functions vs Gain functions: games in explicit form

As mentioned in the Introduction, a major innovation that we propose is the use
of gain functions instead of utility functions. In this section we first elaborate on
the reasons behind this choice that have been sketched in the Introduction and then
we investigate the theoretical consequences of such a choice. First recall the classical
definition of a game in normal form.

Definition 2.1. A finite game in strategic or normal form is given by the following
data:

- a finite set of players \( P = \{1, 2, \ldots, N\} \);
- for each player \( i \in P \), a finite set of strategies \( S_i \);
- for each player \( i \in P \), a preference relation \( \preceq_i \) on \( S := S_1 \times \ldots \times S_N \).

It is frequently convenient (and very often included in the definition of a game) to
specify the players’ preferences by giving real-valued utility functions \( u_i : S \to \mathbb{R} \)
that represent them. The definition and the use of utility functions relies in Morgenstern
and von Neumann’s expected utility theory \cite{Mo-vN47}, where, to avoid problems such as risk
aversion, they assumed that players’ utility functions contain all relevant information
about the players preferences over strategy profiles. In this way, Nash was then able
to formalize Bernoulli’s principle that each player attempts to maximize her expected

\footnote{Joseph Halpern communicated to the author that he and Rafael Pass have independently introduced
super-dominated strategies (under the name minimax dominated strategies) in their forthcoming paper \cite{Ha-Pa13}.}
utility \cite{Be738} given that the other players attempt to do the same. The use of utility functions can certainly make the theory much easier, but it is problematic for at least two reasons:

(1) There are many situations where players constantly violate the principles of expected utility theory. The very first of such examples was found by M. Allais in \cite{Al53} and many others are known nowadays (see, for instance, \cite{Ka-Tv00} and \cite{St00} for a large set of examples). For the sake of completeness, we briefly describe one of these experiments (see \cite{Ka-Tv79}, Problems 3 and 4). In this experiment 95 persons were asked to choose between

L1. A lottery where there is a probability of 0.80 to win 4000 and 0.20 to win nothing.
L2. A certain gain of 3000.

An expected utility maximizer would choose the lottery L1. However, Kahneman and Tversky reported that 80 per cent of the individuals chose the certain gain. The same 95 persons were then asked to choose between

L1’. A lottery with a 0.20 chance of winning 4000 and 0.80 of winning nothing.
L2’. A lottery with a 0.25 chance of winning 3000 and 0.75 of winning nothing.

This time 65 per cent of the subjects chose the lottery L1’, which is also the lottery maximizing expected utility. These two results contradict the so-called substitution axiom in expected utility theory and show how people can behave as expected utility maximizers or not depending on the particular situation they are facing.

An even more dramatic observation is that the evidence suggests that decision makers weight probabilities in a non-linear manner, whereas expected utility theory postulates that they weight probabilities linearly. Consider, for instance, the following example from \cite{Ka-Tv79}, p.283. Suppose that one is compelled to play Russian roulette. One would be willing to pay much more to reduce the number of bullets from one to zero than from four to three. However, in each case, the reduction in probability of a bullet ring is 1/6 and, so, under expected utility theory, the decision maker should be willing to pay the same amount. One possible explanation is that decision makers do not weight probabilities in a linear manner as postulated by expected utility theory.

(2) The application of expected utility theory to game theory leads to the following conceptual problem. Classical game theory makes predictions using expected utility theory but real-life games are always expressed by gain functions, that are, functions whose values represent the quantity of some good which is won or lost by the players. The translation of gain functions to utility functions is an highly non-trivial problem to solve without falling into circular definitions. Consequently, a theory making use of utility functions risks seriously not to satisfy Popper’s falsifiability principle \cite{Po59}.

Problems like the ones in (1) have been now overcome in decision theory thanks to the celebrated prospect theory \cite{Ka-Tv79} and cumulative prospect theory \cite{Tv-Ka92}. Problems like the ones in (2), together with (1), have been recently used as a motivation to replace utility functions with gain functions and then apply cumulative prospect
theory also in game theory [Me-Ri10]. Nevertheless, the authors of [Me-Ri10] have undervalued the fact that passing from utility functions to gain functions may give rise to new phenomena. Indeed, while utility functions were supposed to contain all relevant information about the players preferences, gain functions do not contain such information. They must be taken into account separately. As we will remind in Section 7, risk aversion is taken into account by cumulative prospect theory. Among the remaining relevant information there are (at least) two deserving particular attention:

**Altruism.** A player may prefer to renounce to part of her gain in order to favor another player.

**Perception of gains.** Two different players may have different perceptions about the same amount of gain.

To define formally a game in terms of gain functions, we introduce a unit of measurement \( g \) (typically one dollar, one euro ...) and postulate that to every action profiles \( s \in S \) and to every player \( i \in P \) is associated a quantity \( g_i(s) \) of \( g \) which is lost or won by player \( i \) when the strategy profile \( s \) is played. We assume that the unit of measurement (e.g., the currency) is common. The losses are expressed by negative integers and the wins by positive integers, so that \( g_i(s) = 2 \) will mean, for instance, that, if the strategy profile \( s \) is played, then player \( i \) wins two units of the good \( g \); analogously, \( g_i(s) = -3 \) will mean that, if the strategy profile \( s \) is played, then player \( i \) loses three units of the good \( g \).

Using the unit of measurement, we can take into account altruism and perception of gains as follows. Altruism can be described through a number representing the units of the good \( g \) that player \( i \) is available to renounce in order to favor player \( j \) of one unit of the good. Therefore, to every pair \((i,j), i \neq j\), there will be associated a non-negative real number \( a_{ij} \). The extremal situation \( a_{ij} = 0 \) thus will mean that player \( i \) is selfish against player \( j \). To capture perception of money, we assume that to each player \( i \in P \) is associated a function \( f_i : \{(x,y) \in \mathbb{R}^2 : x \geq y\} \rightarrow [0,\infty) \) whose role is to quantify how much player \( i \) disappreciates to renounce a gain of \( x \) and accept a gain of \( y \). The following are then natural requirements:

- \( f_i \) is continuous,
- if \( x > y \), then \( f_i(x,y) > 0 \),
- if \( x = y \), then \( f_i(x,y) = 0 \),
- for any fixed \( x \), \( f_i(x,\cdot) \) is strictly decreasing and strictly convex,
- for any fixed \( y \), \( f_i(\cdot,y) \) is strictly increasing and strictly concave.

The requirement that \( f_i(x,\cdot) \) is strictly decreasing and strictly convex has the interpretation that the same difference of gains is perceived to be smaller if the gains are higher. A similar interpretation holds for the requirement that \( f_i(\cdot,y) \) is strictly increasing and strictly concave.

Therefore, we are led to study the following object.

**Definition 2.2.** A finite game in explicit form \( G = G(P, S, g, a, f) \) is given by the following data:

- a finite set of players \( P = \{1, 2, \ldots, N\} \);
- for each player \( i \in P \), a finite set of strategies \( S_i \);
- a **good** \( g \), which plays the role of a unit of measurement;
- for each player \( i \in P \), a function \( g_i : S_1 \times \ldots \times S_N \rightarrow \mathbb{Z} \), called **gain function**;
- for each pair of players \((i,j), i \neq j\), an **altruism parameter** \( a_{ij} \geq 0 \);
For each player \(i \in P\), a fairness function \(f_i : \{(x, y) \in \mathbb{R}^2 : x \geq y\} \to \mathbb{R}\) verifying the properties above.

The terminology explicit puts in evidence the fact that we must take into account explicitly all parameters that are usually considered implicit in the definition of utility functions. We are not saying that there are only three such parameters (altruism, the functions \(f_i\), and risk aversion) and this is indeed the first of a long series of points of the theory deserving more attention in future researches.

The purpose of the paper is to define a solution concept for games in explicit form taking into account altruism and cooperation and using cumulative prospect theory instead of expected utility theory. Nevertheless, we will see that

- in most cases the use of cumulative prospect theory instead of expected utility theory can change predictions only quantitatively and not qualitatively;
- in most cases the altruism parameters do not play any active role, since there are no players having a strategy which give a certain disadvantage to other players.

Consequently, we prefer to introduce the cooperative equilibrium in two steps. In the first one we keep expected utility theory and we do not use the altruism parameters. The aim of the first step is only to formalize the principle of cooperation. We show that already this cooperative equilibrium under expected utility theory and without altruism parameters can explain experimental data satisfactorily well. In Section 6 we discuss some examples where the cooperative equilibrium under expected utility theory does not perform well because of the use of expected utility theory and we move towards the definition of the cooperative equilibrium under cumulative prospect theory and taking into account altruism.

### 3. An informal sketch of the definition

In this section we describe the cooperative equilibrium (under expected utility theory and without taking into account altruism) starting from an example. The idea is indeed very simple, even though the complete formalization requires a few definitions that will be given in the next section.

Consider the following variant of the Traveler’s dilemma. Two players have the same strategy set \(S_1 = S_2 = \{80, 81, \ldots, 100\}\). The gain functions are

\[
g_1(x, y) = \begin{cases} 
  x + 2, & \text{if } x < y \\
  x, & \text{if } x = y \\
  y - 2, & \text{if } x > y,
\end{cases} \quad \text{and} \quad g_2(x, y) = \begin{cases} 
  y + 2, & \text{if } x > y \\
  y, & \text{if } x = y \\
  x - 2, & \text{if } x < y.
\end{cases}
\]

The usual backward induction implies that \((80, 80)\) is the unique Nash equilibrium. Nevertheless, numerous experimental studies reject this predictions and show that humans play significantly larger strategies.

In the cooperative equilibrium, the two players try to forecast how the game would be played if they formed coalitions and then they play according to their best forecast. In this case, there are only two possible coalitions, the selfish coalition \(p_s = (\{1\}, \{2\})\) and the cooperative coalition \(p_c = (\{1, 2\})\). Let us analyze them:

- If the players play according to the selfish coalition, then by definition they do not have any incentive to cooperate and therefore they would play the Nash equilibrium \((80, 80)\). A Nash equilibrium is, by definition, stable, in the sense that no players have incentive to change strategy. Consequently, both players
would get 80 for sure. In this case we say that the value of the selfish coalition is 80.

- Now, let us analyze the cooperative coalition $p_c$. The largest gain for each of the two players, if they play together, is to get 200, that is attained by the profile of strategies $(200, 200)$. Nevertheless, every player knows that the other player may defect and play a smaller strategy and so the value of the cooperative coalition is not 200, but we have to take into account possible deviations. Let us look at the problem from the point of view of player 1. The other player, player 2, may deviate and play either the strategy 199 or the strategy 198 (indeed, both these strategies give at least the same gain as the strategy 200, if the first player is believed to play the strategy 200). In this case, the best that player 2 can obtain is $201 - 200 = 1$. We call this number *incentive to deviate* and it will be denoted by $D_2(p_c)$. Now, if player 2 decides to deviate from the coalition, she or he incurs in a *risk* due to the fact that also player 1 can deviate from the coalition either to follow selfish interest or because player 1 is clever enough to understand that player 2 can deviate from the coalition and then player 1 decides to anticipate this move. The maximal risk is then attained when player 2 deviates to 199 and player 1 anticipates this deviation and play 198. In this case, player 2 would gain $g_2(198, 199) = 196$. So in this case, player 2 would get $200-196=4$ less than what she would have gotten if she had not deviated. In this case, we say that the risk is equal to 4 and we write $R_2(p_c) = 4$. We now interpret the number $\tau_{1,\{2\}}(p_c) = \frac{D_2(p_c)}{D_2(p_c) + R_2(p_c)} = \frac{1}{5}$, as a sort of *prior probability* that player 1 assigns to the event *the second player abandons the coalition*. Consequently, we obtain also a number

$$\tau_{1,\emptyset}(p_c) = 1 - \tau_{1,\{2\}}(p_c),$$

which is interpreted as a *prior probability* that player 1 assigns to the event *nobody abandons the coalition*.

This probability measure is now used to weight the numbers $e_{1,\emptyset}(p_c)$, representing the infimum of gains the player 1 receives if nobody abandons the coalition, and $e_{1,\{2\}}(p_c)$, representing the infimum of gains that player 1 receives if the second player abandons the coalition. Therefore, one has

$$e_{1,\emptyset}(p_c) = 200 \quad \text{and} \quad e_{1,\{2\}}(p_c) = 196,$$

where the second number comes from the fact that the worst that can happen for player 1 if the second player abandons the coalition and the first players does not abandon the coalition is in correspondence of the profile of strategies $(200, 198)$ which gives a gain 196 to the first player. Taking the average we obtain the value of the cooperative coalition for player 1

$$v_1(p_c) = 200 \cdot \frac{4}{5} + 196 \cdot \frac{1}{5} \sim 199.$$

By symmetry one has $v_2(p_s) = v_1(p_s) = v(p_s)$ and $v_2(p_c) = v_1(p_c) = v(p_c)$. So one has $v(p_s) < v(p_c)$ and then the cooperative equilibrium predicts that the agents *play according* to the cooperative coalition, since it gives a better forecast. The meaning of the word *play according* has to be clarified. Indeed, since the profile $(200, 200)$ is not stable, we cannot expect that the player play for sure this profile of strategies. What
we do is to interpret the values \( v_i(p_c) \) as a sort of common beliefs: players simply keep only the profiles of strategies \( \sigma = (\sigma_1, \sigma_2) \) such that \( g_1(\sigma) \geq v_1(p_c) \) and \( g_2(\sigma) \geq v_2(p_c) \).

Computing the Nash equilibrium in this induced game will give the cooperative equilibrium of the game that, in this case, is a mixed strategy which is supported near 199.

The purpose of the next section is to formalize the idea that we have just described. Even though this idea is very simple, it will require the whole section 4 because of the following technical problems:

1. In the particular example that we have just described, the cooperative coalition has a unique natural solution, which is (200, 200). In general this will not happen and we should take into account that one solution can be less fair than another. For instance, the cooperative coalition in Nash bargaining problem has many solutions, but intuitively only the (50,50) solution is fair.

2. The definition of deviation and risk is intuitively very simple, but the general mathematical formalization is not very straightforward.

4. **The cooperative equilibrium under expected utility theory**

Let \( G = G(P, S, g, a, f) \) be a finite\(^7\) game in explicit form. As usual, to make notation lighter, we denote \( S_i \) the cartesian product of all the \( S_j \)’s but \( S_i \). Let \( \mathcal{P}(X) \) be the set of probability measures on the finite set \( X \). If \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{P}(S_1) \times \cdots \times \mathcal{P}(S_N) \), we denote by \( \sigma_{-i} \) the \((N-1)\)-dimensional vector of measures \( (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N) \) and, as usual in expected utility theory, we set

\[
g_j(\sigma_i, \sigma_{-i}) = g_j(\sigma) := \sum_{(s_1, \ldots, s_N) \in S} g_j(s_1, \ldots, s_N)\sigma_1(s_1) \cdots \sigma_N(s_N).
\]

Conversely, if \( \sigma_i \in \mathcal{P}(S_i) \), for all \( i \in P \), the notation \( g_j(\sigma_i, \sigma_{-i}) \) simply stands for the number \( g_j(\sigma_1, \ldots, \sigma_N) \).

The main idea behind our definition is the principle of cooperation, that is, players try to forecast how the game would be played if they formed coalitions and then they play according to their best forecast. Borrowing a well known terminology from the literature about coalition formation (cf. \cite{Ra08}), we give the following

**Definition 4.1.** A coalition structure is a partition \( p = (p_1, \ldots, p_k) \) of the player set \( P \); that is, the \( p_\alpha \)'s are subsets of \( P \) such that \( p_\alpha \cap p_\beta = \emptyset \), for all \( \alpha \neq \beta \), and \( \bigcup p_\alpha = P \).

As mentioned in the Introduction, the idea is that each player \( i \in P \) assigns a value to each coalition structure \( p \) and then plays according to the coalition with highest value. As described in Section \ref{subsection:coalition}, the idea to define the value of a coalition structure \( p \) for player \( i \) is to take an average of the following kind. Let \( J \subseteq P \setminus \{i\} \) and suppose we have defined a number \( r_{i,J}(p) \) describing the probability that players in \( J \) abandon the coalition and a number \( c_{i,J}(p) \) describing the infimum of possible gains of player \( i \) when

\(^7\)It is well known that the study of infinite games can be very subtle. For instance, there is large consensus that, at least when the strategy sets do not have a natural structure of a standard Borel space, one must allow also purely finitely additive probability measures as mixed strategies, leading to the problem that even the mixed extension of the utility functions is not uniquely defined \cite{Ma97,ST05,Ca-Mo12,Ca-Sc12}. In this first stage of the research we want to avoid all these technical issues and we focus our attention only to finite games.
players in $J$ abandon the coalition. Then we (would) define

$$v_i(p) = \sum_{J \subseteq P \setminus \{i\}} e_i(J(p)) \tau_i(J(p)).$$  \hspace{1cm} (1)

Our aim is to give a reasonable definition for the numbers $e_i(J(p))$ and $\tau_i(J(p))$. Before doing that we need to understand what kind of strategies agree with the coalition. Indeed, as mentioned in Section 3, if $p \neq (\{1\}, \ldots, \{N\})$ is not the selfish coalition, some profiles of strategies might not be acceptable by the players in the same coalition because they do not share the payoff in a fair way between the players belonging to a same coalition $p_\alpha$. We can define a notion of fairness making use of the fairness functions $f_i$. First observe that the hypothesis of working with gain functions expressed using the same unit of measurement for all players allows us to sum the gains of different players and, consequently, we can say that a coalition structure $p = (p_1, \ldots, p_k)$ generates a game with $k$ players as follows. The players are the sets $p_\alpha$ in the partition, the pure strategy set of $p_\alpha$ is $\prod_{i \in p_\alpha} S_i$, and the utility function of player $p_\alpha$ is

$$g_{p_\alpha}(s_1, \ldots, s_N) = \sum_{i \in p_\alpha} g_i(s_1, \ldots, s_N)$$  \hspace{1cm} (2)

This game, that we denote by $G_p$, has a non-empty set of Nash equilibria that we denote by $\text{Nash}(G_p)$. Since the players in the same $p_\alpha$ are ideally cooperating, not all Nash equilibria are acceptable, but only the ones that distribute the payoff of the coalition $p_\alpha$ as fairly as possible among the players belonging to $p_\alpha$.

To define the subset of fair of acceptable equilibria, fix $i \in P$ and consider the restricted function $\tilde{g}_i = g_i|_{\text{Nash}(G_p)} : \text{Nash}(G_p) \to \mathbb{R}$. Since $\text{Nash}(G_p)$ is compact and $g_i$ is continuous, we can find $\sigma_i \in \text{Nash}(G_p)$ maximizing $\tilde{g}_i$.

**Definition 4.2.** The disagreement in playing the profile of strategy $\sigma \in \text{Nash}(G_p)$ for the coalition $p_\alpha$ is the number

$$\text{Dis}_{p_\alpha}(\sigma) = \sum_{i \in p_\alpha} f_i(g_i(\sigma_i), g_i(\sigma))$$

Recalling that the number $f_i(x, y)$ represent how much player $i$ disappreciates to renounce to a gain of $x$ and accept a gain of $y \leq x$, we obtain that, in order to have a fair distribution of the payoff among the players in the coalition $p_\alpha$, the disagreement $\text{Dis}_{p_\alpha}$ must be minimized.

**Definition 4.3.** The Nash equilibrium $\sigma \in \text{Nash}(G_p)$ is acceptable or fair for the coalition $p_\alpha$, if $\sigma$ minimizes $\text{Dis}_{p_\alpha}(\sigma)$.

Since the set of Nash equilibria of a finite game is compact and since the functions $f_i$ are continuous, it follows that the set of acceptable equilibria is non-empty and compact.

Let us say explicitly that this is the unique point where we use the functions $f_i$. It follows, that, for a game $G$ such that every game $G_p$ has a unique Nash equilibrium, the cooperative equilibrium does not depend on the functions $f_i$.

The importance of the hypotheses about strict convexity in the second variable and strict concavity in the first variable of the functions $f_i$ should be now clear and is however described in the first of the following series of examples.

---

8If $p = (P)$ is the large coalition, then $G_p$ is a one-player game, whose Nash equilibria are all probability measures supported on the set of strategies maximizing the payoff function.
Example 4.4. Consider a finite version of Nash’s bargaining problem \([Na50b]\) where two persons have the same strategy set \(S_1 = S_2 = S = \{0, 1, \ldots, 100\}\) and the gain functions are as follows:

\[
g_1(x, y) = \begin{cases} x, & \text{if } x + y \leq 100 \\ 0, & \text{if } x + y > 100 \end{cases}
\]

and

\[
g_2(x, y) = \begin{cases} y, & \text{if } x + y \leq 100 \\ 0, & \text{if } x + y > 100 \end{cases}
\]

As well known, this game has attracted attention from game theorists since, despite having a lot of pure Nash equilibria, only one is intuitively natural. Indeed, many papers have been devoted to select this natural equilibrium adding axioms (see \([Na50b]\), \([Ka-Sm75]\), and \([Ka77]\)) or using different solution concepts (see \([Ha-Ro10]\) and \([Ha-Pa12]\)).

Assume that the two players have the same perception of money, that is \(f_1 = f_2\).

Consider the cooperative coalition \(p_c = (\{1, 2\})\) describing cooperation between the two players. The game \(G_{p_c}\) is a one-player game whose Nash equilibria are all pairs \((x, 100 - x)\), \(x \in S_1\), and all probability measures on \(S_1 \times S_2\) supported on such pairs of strategies. Despite having all these Nash equilibria, the unique acceptable equilibrium for the game coalition is \((50, 50)\). Indeed, one has

\[
\text{Dis}_{p_c}(50, 50) = f(100, 50) + f(100, 50)
\]

\[
= f\left(\frac{1}{2} \cdot 100 + \frac{1}{2} \cdot 0\right) + f\left(\frac{1}{2} \cdot 100 + \frac{1}{2} \cdot 0\right)
\]

\[
< \frac{1}{2} f(100, 100) + \frac{1}{2} f(100, 0) + \frac{1}{2} f(100, 100) + \frac{1}{2} f(100, 0)
\]

\[
= f(100, 100) + f(100, 0)
\]

\[
= \text{Dis}_{p_c}(100, 0).
\]

Analogously, one gets \(\text{Dis}_{p_c}(50, 50) < \text{Dis}_{p_c}(x, 100 - x)\), for all \(x \in \{0, 1, \ldots, 100\}\), \(x \neq 50\). Consequently, \((50, 50)\) is the unique acceptable equilibrium for the cooperative coalition \(p_c\).

Now let \(p_s = (\{1\}, \{2\})\) be the selfish coalition. Then the unique acceptable equilibrium for player 1 is \((100, 0)\) and the unique acceptable Nash equilibria for player 2 is \((0, 100)\).

Example 4.5. As second example, we consider the Prisoner’s Dilemma. As well known, this famous game were originally introduced by Flood in \([Fl52]\), where he reported on a series of experiments, one of which, now known as Prisoner’s Dilemma, was conducted in 1950. Even though Flood’s report is seriously questionable, as also observed by Nash himself (cf. \([Fl52]\), pp. 24-25), it probably represents the first evidence that humans tend to cooperate in the Prisoner’s Dilemma. This evidence has been confirmed in \([Co-DJ-Fo-Ro96]\), where the authors observed cooperation even in one-shot version of the Prisoner’s dilemma.

Here we consider a parametrized version of the Prisoner’s Dilemma, as follows. Two persons have the same strategy set \(S_1 = S_2 = \{C, D\}\), where C stands for cooperate and D stands for defect. Let \(\mu > 0\), denote by \(G(\mu)\) the game described by the following gains:

\[
\begin{array}{cc}
C & D \\
\hline
1 + \mu & 1 + \mu \\
2 + \mu & 0 \\
2 + \mu & 1 + 1
\end{array}
\]

Therefore, the parameter \(\mu\) plays the role of a reward for cooperating. The intuition, motivated by similar experiments conducted on the Traveler’s Dilemma (cf. Example
suggests that humans should play the selfish strategy D for very small values of μ and tend to cooperate for very large values of μ. We will see in Example 5.8 that this is indeed what the cooperative equilibrium predicts. For now, let us just compute the acceptable Nash equilibria for the two partitions of \( P = \{1, 2\} \). Let \( p_c = (\{1\}, \{2\}) \) be the cooperative coalition, describing cooperation between the players. In this case we obtain a one-player game with gains:

\[
\begin{align*}
g_{p_c}(C,C) &= 2 + 2\mu \\
g_{p_c}(C,D) &= 2 + \mu \\
g_{p_c}(D,C) &= 2 + \mu \\
g_{p_c}(D,D) &= 2
\end{align*}
\]

whose unique Nash equilibrium (i.e., the profile of strategies maximizing the payoff) is the cooperative profile of strategies (C,C). Uniqueness implies that this equilibrium must be acceptable independently of the \( f_i \)'s. On the other hand, the selfish coalition profile \( p_s = (\{1\}, \{2\}) \) generates the original game, whose unique equilibrium is, as well known, the defecting profile of strategies (D,D). Also in this case, uniqueness implies that this equilibrium must be acceptable.

**Example 4.6.** Finally, we consider the Traveler’s Dilemma. This game was introduced by Basu in [Ba94] with the purpose to construct a game where Nash equilibrium makes unreasonable predictions. Basu’s intuition was indeed confirmed in a series of experiments reported in [Ca-Go-Go-Ho99], where the authors considered a variant of Basu’s original game as follows. Fix a parameter \( b \in \{2, 3, \ldots, 80\} \), two players have the same strategy set \( S_1 = S_2 = \{80, 81, \ldots, 200\} \) and payoffs:

\[
\begin{align*}
g_1(x, y) &= \begin{cases} 
  x + b, & \text{if } x < y \\
  x, & \text{if } x = y \\
  y - b, & \text{if } x > y,
\end{cases} \\
\text{and } g_2(x, y) &= \begin{cases} 
  y + b, & \text{if } x > y \\
  y, & \text{if } x = y \\
  x - b, & \text{if } x < y.
\end{cases}
\end{align*}
\]

This game has a unique Nash equilibrium, which is (80,80). Nevertheless, in the experiments reported in [Ca-Go-Go-Ho99], it has been observed that humans tend to cooperate (i.e., play strategies close to (200,200)) for small values of \( b \) and tend to be selfish (i.e., play strategies close to the Nash equilibrium (80,80)) for large values of \( b \). This observation was lately confirmed by other experiments (see [Be-Ca-Na05] and [Ba-Be-St11]). This is indeed what the cooperative equilibrium predicts, as we will see in Example 5.1. For now, let us just compute the sets of acceptable equilibria for all partitions of \( P = \{1, 2\} \). Let \( p_c = (\{1\}, \{2\}) \) be the cooperative coalition, describing cooperation between the players. In this case we obtain a one-player game whose unique Nash equilibrium is attained by the cooperative profile of strategies (200,200). Uniqueness implies that this equilibrium must be acceptable. On the other hand, the selfish partition \( p_s = (\{1\}, \{2\}) \) gives rise to the unique Nash equilibrium of the game, which is (80,80). Also in this case, uniqueness implies that this equilibrium must be acceptable.

Coming back to the description of the theory, we have gotten, for all partitions \( p \) of the player set \( P \) and for all sets \( p_\alpha \) of the partition, a (compact) set of acceptable equilibria \( \text{Acc}_{p_\alpha}(G_p) \) for the coalition \( p_\alpha \) inside the coalition structure \( p \). Now we can work out a definition of the numbers \( e_{i,J}(p) \) and \( \tau_{J}(p) \).

**Definition of the numbers \( \tau_{i,J}(p) \).** We recall that the number \( \tau_{i,J}(p) \) represents the probability that players in \( J \) deviate from the coalition \( p \). Consequently, it is enough to define the numbers \( \tau_{i,J}(p) \) when \( J = \{\} \) contains only one element. The other numbers can be indeed reconstructed assuming that the events “player \( j \) deviates from \( p \)" and
“player $k$ deviates from $p^*$” are independent. Therefore, fix $j \in P$, with $j \neq i$. The definition of $\tau_{i,j}(p)$ is intuitively very simple. It will be a ratio

$$\tau_{i,j}(p) = \frac{D_j(p)}{D_j(p) + R_j(p)},$$

where:

- the number $D_j(p)$ represents the incentive for player $j$ to abandon the coalition structure $p$, that is, the maximal gain that player $j$ can get leaving the coalition;
- the number $R_j(p)$ represents the risk that player $j$ leaves the coalition structure $p$, that is, the maximal loss that player $j$ can get leaving the coalition, assuming that also other players can leave the coalition.

To make this idea formal, first define

$$\tilde{M}(p_\alpha, p) := \{\sigma \in \text{Acc}_{p_\alpha}(G_p) : g_{p_\alpha}^\sigma(\sigma) \text{ is maximal}\}. \quad (3)$$

Now let us first fix a piece of notation. Let $\pi_j : \mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N) \rightarrow \mathcal{P}(S_j)$ be the canonical projection. We may reconstruct an element $\sigma \in \mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$, through its projections and we write formally $\sigma = \bigotimes_{j=1}^N \pi_j(\sigma)$. Set

$$M(p_\alpha, p) := \left\{ \bigotimes_{i \in p_\alpha} \pi_i(\sigma) : \sigma \in \tilde{M}(p_\alpha, p) \right\}. \quad (4)$$

**Definition 4.7.** Let $\sigma \in \mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$ be a profile of mixed strategies and $\sigma_k' \in \mathcal{P}(S_k)$. We say that $\sigma_k'$ is a $k$-deviation from $\sigma$ if $g_k(\sigma_k', \sigma_{-k}) \geq g_k(\sigma)$.

Let

$$\text{Dev}_j(p) := \left\{ (\sigma, \sigma_j') \in \bigotimes_{\alpha=1}^k M(p_\alpha, p) \times \mathcal{P}(S_j) : g_j(\sigma_j', \sigma_{-j}) \geq g_j(\sigma) \right\}. \quad (5)$$

**Definition 4.8.** The incentive for player $j$ to deviate from the coalition structure $p$ is

$$D_j(p) := \max \left\{ g_j(\sigma_j', \sigma_{-j}) - g_j(\sigma) : (\sigma, \sigma_j') \in \text{Dev}_j(p) \right\}. \quad (6)$$

Observe that $D_j(p)$ is attained since the sets $M(p_\alpha, p)$, $\mathcal{P}(S_j)$, and, consequently, $\text{Dev}_j(p)$ are compact.

If $D_j(p) = 0$, then $j$ does not gain anything leaving the coalition and therefore $j$ does not have any incentives to abandon the coalition structure $p$. If it is the case, we simply define $\tau_{i,j}(p) = 0$.

Consider now the more interesting case $D_j(p) > 0$, where player $j$ has an actual incentive to deviate from the coalition. If $j$ decides to leave a coalition, it may happen that she loses part of her gain if other players decide to abandon the coalition either to follow selfish interests or to answer player $j$’s defection. To quantify this risk, we first introduce some notation. Let $(\sigma, \sigma_j') \in \text{Dev}_j(p)$ such that $D_j(p)$ is attained. Call $T(\sigma, \sigma_j')$ the set of $\sigma_{-j} \in \bigotimes_{i \neq j} \mathcal{P}(S_i)$ such that

- $g_j(\sigma) - g_j(\sigma_j', \sigma_{-j}) > 0$,
- there is $k \in P \setminus \{j\}$ such that $\pi_k(\sigma_{-j})$ is a $k$-deviation from either $\sigma$ or $\sigma_{-j}$. 

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9This assumption appears natural in a context where the players are not allowed to exchange information.
Thus we quantify the risk by
\[ R_j(p) := \sup \{ g_j(\sigma) - g_j(\sigma', \sigma_{-j}) \}, \tag{7} \]
where the supremum is taken over all
1. \((\sigma, \sigma'_j) \in \text{Dev}_j(p)\) such that \(D_j(p)\) is attained,
2. \(\sigma'_{-j} \in \mathcal{T}(\sigma, \sigma'_j)\).

The requirement (2) is motivated by the fact that if player \(j\) believes that she can leave the coalition to follow selfish interests, then she must take into account that also other players may deviate from the coalition either to follow selfish interests or because they are clever enough to anticipate player \(j\)’s defection. This can obstruct player \(j\)’s deviation, if another player deviation causes a loss to player \(j\).

**Definition 4.9.** The prior probability that player \(j\) deviates from the coalition structure \(p\) is
\[ \tau_j(p) := \frac{D_j(p)}{D_j(p) + R_j(p)}. \]

The terminology prior wants to clarify the fact that the event “player \(j\) abandons the coalition” is not measureable in any absolute and meaningful sense. The prior probability is a sort of measure a priori of this event knowing only mathematically measurable information, as monetary incentive and monetary risk.

As said earlier, we can now compute all remaining probabilities \(\tau_{i,J}(p)\) assuming that the events “player \(j\) deviates from \(p\)” and “player \(k\) deviates from \(p\)” are independent. In particular, \(\tau_{i,\emptyset}(p)\) will represent the probability that none of the players other than \(i\) deviates from the coalition structure.

**Definition of the numbers** \(e_{i,J}(p)\). The numbers \(e_{i,J}(p)\) represent the infimum of gains of player \(i\) when the players in \(J\) decide to deviate from the coalition. Therefore, the definition of these numbers is very straightforward. Let \(J \subseteq P \setminus \{i\}\), we first define the set
\[ \text{Dev}_J(p) := \left\{ (\sigma, \sigma'_J) \in \bigotimes_{a=1}^k M(p_a, p) \times \bigotimes_{j \in J} \mathcal{P}(S_j) : \exists j \in J : g_j(\pi_j(\sigma'_J), \sigma_{-j}) \geq g_j(\sigma) \right\}. \]

Then we define
\[ e_{i,J}(p) := \inf\{g_i(\sigma'_J, \sigma_{-j}) : (\sigma, \sigma'_J) \in \text{Dev}_J(p)\}. \]

**Definition 4.10.** The value of the coalition structure \(p\) for player \(i\) is
\[ v_i(p) = \sum_{J \subseteq P \setminus \{i\}} e_{i,J}(p) \tau_{i,J}(p). \tag{8} \]

We stress that at this first stage of the research we cannot say that this formula is eventually the right way to compute the value of a coalition structure. It just seems a fairly natural way and, as we will show in Section 5, it meets experimental data satisfactorily well. However, it is likely that future researches, possibly supported by suitable experiments, will suggest to use of a different formula. For instance, we will describe in Example 6.2 that it is possible that the deviation \(D_j(p)\) should be computed
taking into account not only deviation to achieve higher gains, but also to get a safe gain.

Now, in an exact theory, player $i$ is assumed not to make mistakes in the computations and so, using the principle of cooperation, she will play according to some $p$ which maximises the value function $v_i$. It remains to understand the meaning of playing according with a coalition structure $p$. Indeed, we cannot expect that player $i$ will play surely according to an acceptable Nash equilibrium of $G_p$, since she knows that other players may deviate from the coalition. What we can do is to use the numbers $v_i(p)$ to define a sort of beliefs.

**Definition 4.11.** Let $A \subseteq \mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$. The subgame induced by $A$ is the game whose set of mixed strategies of player $i$ is the closed convex hull in $\mathcal{P}(S_i)$ of the projection set $\pi_i(A)$.

Therefore, a subgame induced by a set $A$ is not, strictly speaking, a game, since in general the set of mixed strategies of player $i$ cannot be described as the convex hull of a set of pure strategies which is a subset of $S_i$. In the induced game only particular mixed strategies are allowed, which, as said earlier, correspond to some sort of beliefs. Observe that, since the set of allowed mixed strategies is convex and compact, we can formally find a Nash equilibrium of an induced game. Indeed, Nash’s proof of existence of equilibria does not really use the fact that the utility functions are defined on $\mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$, but only that they are defined on a convex and compact subset of $\mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$.

Let $\text{Ind}(G, p)$ be the subgame induced by the set of strategies $\sigma \in \mathcal{P}(S_1) \times \ldots \times \mathcal{P}(S_N)$ such that $g_i(\sigma) \geq v_i(p)$, for all $i \in P$. Observe that the induced game is not empty, since $v_i(p)$ is a convex combinations of infima of values attained by the gain function.

**Definition 4.12. (Exact cooperative equilibrium)** An exact cooperative equilibrium is one where player $i$ plays a Nash equilibrium of the subgame $\text{Ind}(G, p)$ where $p$ maximizes $v_i(p)$.

One could define a quantal cooperative equilibrium, declaring that player $i$ plays with probability $e^{\lambda v_i(p)}/\sum p e^{\lambda v_i(p)}$ according to the quantal response equilibrium or the quantal level-k theory applied to $\text{Ind}(G, p)$. At this first stage of the research, we are not interesting in such refinements, that could be useful in future and deeper analysis (cf. Examples 6.2 and 9.8).

5. Examples

In this section we apply the cooperative equilibrium (under expected utility theory and without using altruism) to some examples. The results we obtain are very much encouraging, since the predictions of the cooperative equilibrium are always close to the experimental data. We present also two examples where the cooperative equilibrium makes new predictions, completely different from all standard theories. These new predictions are partially supported by experimental data, but we do not have enough precise data to say that they are strongly confirmed.

**Example 5.1.** Let $G^{(b)}$ be the parametrized Traveler’s Dilemma in Example 4.6 with bonus-penalty equal to $b$. Let $p_c = \{1, 2\}$ be the cooperative coalition. We recall that

---

10Observe that this is well defined also in case of multiple $p$'s maximizing $v_i(p)$, since the induced games $\text{Ind}(G, p)$ and $\text{Ind}(G, p')$ are the same, if $p, p'$ are both maximizers.
in Example 4.6 we have shown that the profile of strategies $(200, 200)$ is the unique acceptable equilibrium for $p_c$. To compute the values, let $i = 1$ (the case $i = 2$ is the same, by symmetry). One has $D_2(p_c) = b - 1$, corresponding to the strategy profile $(200, 199)$. Corresponding to this deviation of player 2, the best deviation for player 1 is to play the strategy 198, which gives $g_2(200, 200) - g_2(198, 199) = 2 + b$. On the other hand, corresponding to the strategy profile $(200, 200)$, the best deviation for player 1 is to play the strategy 199, which gives $u_2(200, 200) - u_2(199, 100) = 1 + b$. Therefore, $R_2(p_c) = \max\{1 + b, 2 + b\} = 2 + b$. Consequently, we have

$$\tau_1(2)(p_c) = \frac{b - 1}{2b + 1} \quad \text{and} \quad \tau_1(0)(p_c) = \frac{b + 2}{2b + 1}.$$

Now, $e_1(2) = 200 - 2b$, corresponding to the profile of strategy $(200, 200 - b)$, and $e_1(0)(p_c) = 200$. Consequently, setting $v_1(p_c) = v_2(p_c) =: v(p_c)$, we have

$$v(p_c) = 200 \cdot \frac{b + 2}{2b + 1} + (200 - 2b) \cdot \frac{b - 1}{2b + 1}.$$ 

On the other hand, the selfish coalition structure $p_s = (\{1\}, \{2\})$ has value

$$v_1(p_s) = v_2(p_s) = 80,$$

since there are no possible deviations from a Nash equilibrium. Therefore, for small values of $b$, one has $v(p_c) > v(p_s)$ and the cooperative equilibrium predicts that agents play according to the cooperative coalition; for large values of $b$, one has $v(p_s) > v(p_c)$ and then the cooperative equilibrium predicts that agents play the Nash equilibrium.

Let us make a comparison between the predictions of the cooperative equilibrium and the experimental data.

- For $b = 2$ and $S_1 = S_2 = \{2, 3, \ldots, 100\}$, it has been reported in [Be-Ca-Na05] that most of subjects (31 out of 35) chose a number between 96 and 100 and the strategy which had the highest payoff was $s = 97$. In our case, we obtain

$$v(p_c) = 100 \cdot \frac{b + 2}{2b + 1} + 96 \cdot \frac{b - 1}{2b + 1} = 98.67.$$

Consequently, the cooperative equilibrium is supported between 98 and 99, which is very close to the experimental data.

- For $b = 2$ and $S_1 = S_2 = \{180, 181, \ldots, 300\}$, it has been reported in [Go-Ho01] that about 80 per cent of the subjects submitted a strategy between 290 and 300, with an average of 295. In our case, we obtain

$$v(p_c) = 300 \cdot \frac{b + 2}{2b + 1} + 290 \cdot \frac{b - 1}{2b + 1} = 296.35.$$

Consequently, the cooperative equilibrium is supported between 98 and 99, which is very close to the experimental data.

Again for $b = 5$, but also for $b = 10$ we have experimental data collected in [Ca-Go-Go-Ho99], Table 3. These data have been collected making an experiment on ten periods. Since the cooperative equilibrium is defined for one-shot games, as usual we use the data collected for the last period. Unfortunately, these data have not been reported in [Ca-Go-Go-Ho99], but they reported the average claims for the last three periods. We believe that this data could be used as a satisfactorily precise information about how the players would be playing the one-shot game.
• For $b = 5$ and $S_1 = S_2 = \{80, 81, \ldots, 200\}$, the average observed claim was 195. The value of the cooperative coalition, for $b = 5$, is

$$v(p_c) = 200 \cdot \frac{b + 2}{2b + 1} + 190 \cdot \frac{b - 1}{2b + 1} \sim 196.36.$$ 

Consequently, the cooperative equilibrium is supported between 196 and 197, which is very close to what observed.

• For $b = 10$ and $S_1 = S_2 = \{80, 81, \ldots, 200\}$, the average observed claim was 186. The value of the cooperative coalition, for $b = 10$, is

$$v(p_c) \sim 181.4.$$ 

Consequently, the cooperative equilibrium is supported between 181 and 182, giving a satisfactorily precise predictions.

Finally we make a comparison between the cooperative equilibrium and experimental data for very large values of $b$. In the case $b = 180$ and $S_1 = S_2 = \{180, 181, \ldots, 300\}$ we have experimental data collected on a one-shot Traveler’s dilemma in [Go-Ho01]. For $b = 50$ and $b = 80$ we use again the experimental data collected on the iterated Traveler’s dilemma in [Ca-Go-Go-Ho99]. Nevertheless, this time we believe that we should trust the average first claim more than the average claim in the last periods. Indeed, the penalty is so high that, as soon as one player claims less than the other, this player has a strong incentive to claim very little in the next period. This generates a quick convergence towards Nash equilibrium after a few iterations that was indeed observed in [Ca-Go-Go-Ho99].

• For $b = 50$ and $S_1 = S_2 = \{80, 81, \ldots, 200\}$, it was reported that the average first claim was 155. The value of the cooperative coalition, for $b = 50$, is

$$v(p_c) \sim 151.48.$$ 

Consequently, the cooperative equilibrium is supported between 151 and 152, giving a satisfactorily precise predictions.

• For $b = 80$ and $S_1 = S_2 = \{80, 81, \ldots, 200\}$, it was reported that the average first claim was 120. The value of the cooperative coalition, for $b = 80$, is

$$v(p_c) \sim 121.49.$$ 

Consequently, the cooperative equilibrium is supported between 121 and 122, giving a very precise precise predictions.

• For $b = 180$ and $S_1 = S_2 = \{180, 181, \ldots, 300\}$, it was reported in [Go-Ho01] that about 80 per cent of the subjects played the Nash equilibrium 180. In our case, one easily sees that

$$v(p_c) < v(p_s)$$

Consequently, the cooperative equilibrium reduced to Nash equilibrium and predicts the solution $(180, 180)$. Observe that, for such large values of $b$, iterated regret minimization predicts complete defection for sure\footnote{Iterated regret minimization predicts defection for $b \geq 49$.}. This is one of the cases where cooperative equilibrium makes different predictions from iterated regret minimization and this predictions seems closer to the actual human behavior.
Example 5.2. Let us consider the Bertrand competition. Each of $N$ players simultaneously chooses an integer between 2 and 100. The player who chooses the lowest number gets a dollar amount times the number she bids and the rest of the players get 0. Ties are split among all players who submit the corresponding bid.

The unique Nash equilibrium of this game is to choose 2. Nevertheless, it has been reported in [Du-Gn00] that humans tend to choose larger numbers. It was also observed that the claims tend to get closer to the Nash equilibrium, when the number of players gets larger and larger.

To compute the value of the cooperative coalition $p_c = \{1, \ldots, N\}$ we observe that every player $j$ has incentive $D_j(p_c) = 49$ and risk $R_j(p_c) = 50$. We then obtain

- For $N = 2$, $v_1(p_c) = v_2(p_c) = 50 \cdot \frac{51}{99}$,
- For $N = 4$, one has $v_1(p_c) = \ldots = v_N(p_c) = 50 \cdot \left(1 - 3 \cdot \frac{49}{99} + 3 \cdot \left(\frac{49}{99}\right)^2 - \left(\frac{49}{99}\right)^3\right)$,
- and so forth.

In other words, using the law of total probability, one can easily show that the value of the cooperative coalition converges to 0 very quickly. Consequently, when $N$ increases, the value decreases and the cooperative equilibrium predicts smaller and smaller claims. This matches qualitatively what reported in [Du-Gn00]. Observe that also quantitatively, the predictions of the cooperative equilibrium are satisfactorily precise. Indeed,

- For $N = 2$, the value of the cooperative coalition is approximately 25 and, therefore, the cooperative equilibrium predicts a claim of approximately 50. It was reported that, in the last period of the iterated Bertrand competition conducted in [Du-Gn00], people claimed 49.6.
- For $N = 4$, the value of the cooperative coalition is approximately 6.5 and, therefore, the cooperative equilibrium predicts a claim of approximately 13. It was reported that, in the last period of the iterated Bertrand competition conducted in [Du-Gn00], people claimed 20.5. In this case the quantitative predictions is slightly different but we will show in Section 6 that it gets much more precise using cumulative prospect theory.

Example 5.3. In this example we show that the cooperative equilibrium theory fits an experiment reported by Kahneman, Knetsch and Thaler in [KKT86]. Consider a simple ultimatum game. A proposer and a responder bargain about the distribution of a surplus of fixed size that we suppose normalized to one. The responder’s share is denoted by $s$ and the proposer’s share by $1 - s$. The bargaining rules stipulate that the proposer offers a share $s \in [0, 1]$ to the responder. The responder can accept or reject $s$. In case of acceptance the proposer receives a monetary payoff $1 - s$, while the responder receives $s$. In case of a rejection both players receive a monetary return of zero.

Kahneman, Knetsch and Thaler conducted the following experiment: 115 subjects were asked to say what would be the minimum offer (between 0 and 10 Canadian dollars) that they would accept, if they were responders. The mean answer was between 2.00 and 2.59.

Now, cooperative equilibrium theory predicts that the responder would accept any offer larger than the value of the coalition structure with the largest value. So let us compute the value for the responder of the two coalition structures $p_s$ and $p_c$ assuming that the two players have the same perception of money.
Denote by $A$ and $R$ responder’s actions accept and reject, respectively. As in Nash bargaining problem, we obtain that the cooperative coalition $p_c = \{1, 2\}$ leads to a one-player game $G_{p_c}$ with the unique acceptable equilibrium $(5, A)$. Therefore, we have

$$v_2(p_c) = \frac{5}{2}$$

since the first player can abandon the coalition playing every $s < \frac{5}{2}$, but she risks to lose everything if the second player rejects the offer (observe that $R$ is a 2-deviation to the strategy $s = 0$).

Consequently, cooperative equilibrium theory predicts that the responder would accept any offer larger than 2.5 dollars, which fits the experimental data reported in [KKT86].

**Remark 5.4.** The cooperative equilibrium can predict well also other experimental data collected for the ultimatum game.

Recall that the unique subgame perfect equilibrium of the ultimatum game is to offer $s = 0$. Nevertheless, there are numerous experimental studies which reject this prediction and show that proposers almost always make substantially larger offers. Fehr and Schmidt [Fe-Sc99] explained these observations making use of two parameters $\alpha_i, \beta_i$ for each player. Let us find out what happens using cooperative equilibrium.

Concerning the selfish coalition $p_s = (\{1\}, \{2\})$. One easily sees that

$$v_1(p_s) = v_2(p_s) = 0,$$

in correspondence to the subgame perfect equilibrium $(0, R)$. Concerning the cooperative coalition, we have

$$v_1(p_c) = \frac{1}{2},$$

since the second player has no incentive to abandon the coalition, and

$$v_2(p_c) = \frac{1}{4}$$

as shown in Example 5.3. Consequently, the exact cooperative equilibrium predicts that the proposer offers $s = 0.25$ and the responder accepts. This explains the fact that there are virtually no offer below 0.2 and above 0.5, which was observed in [Fe-Sc99] making a comparison among experimental data collected in [GSS82], [KKT86], [FHSS88], [RPOZ91], [Ca95], [HMSc96], and [SI-Ro97].

So there are some data that can be explained by the cooperative equilibrium. There are also other data that cannot be explained by the cooperative equilibrium. For instance, it was observed that proposer’s offer was very often higher than 0.25 and, in most of the cases, it was between 0.4 and 0.5 (cf. [Fe-Sc99], Table I). This deviation towards cooperation is not predicted by the exact cooperative equilibrium. We believe that this deviation is due to the fact that the game is not simultaneous, fact that allows reasonings of the kind: if the proposer makes an offer too close to 0.25, then the probability of rejection is too high and therefore the proposer prefers to offer a fairer share. This way of reasoning is also justified by the fact that $v_1(p_c) = 0.5$ and $v_2(p_c) = 0$, which implies that the proposer is anyway happy with a gain larger than 0.5.

We now discuss an example that we believe is relevant because it makes predictions that are significantly different from Nash equilibrium. Such predictions are partially confirmed by experimental data, but it would be important to conduct more precise experiments in order to see how humans behave in such a situation.
Example 5.5. Let us consider a public good game as follows. For computational simplicity, we assume that there are only two players, each of which decides on their contribution levels \( x_i \in [0, y] \) to the public good. Each player has an endowment of \( y \). The monetary payoff of player \( i \) is given by

\[
g_i(x_1, x_2) = y - x_i + \alpha(x_1 + x_2),
\]

where \( \frac{1}{2} < \alpha < 1 \) denotes the constant marginal return to the public good \( X = x_1 + x_2 \).

Notice that the unique perfect equilibrium is to choose \( x_i = 0 \). This prediction is confirmed by a number of experimental studies (cf. [Fe-Sc99], Table II). These studies show that a large percentage of players really played the equilibrium \( x_i = 0 \) or close to it. However, the experiments reported in [Fe-Sc99], Table II, concern only relatively small values of \( \alpha \) and one can observe that indeed the percentage of people claiming \( x_i = 0 \) gets slightly smaller as \( \alpha \) increases. We now show that this tendency to abandon the Nash equilibrium to play a more cooperative solution is predicted by the cooperative equilibrium.

Let \( p_c = \{\{1, 2\}\} \) be the cooperative coalition. The unique Nash equilibrium of the game \( G_{p_c} \) is \( (y, y) \) and each of the two players gets \( e_{1, p_c} = 2\alpha y \). Assume \( i = 1 \) (the case \( i = 2 \) is symmetric). Observe that \( D_2(p_c) = y + \alpha y - 2\alpha y = y - \alpha y \). Indeed, the best deviation for player 2 is to play \( x_2 = 0 \), which gives a payoff of \( y + \alpha y \), if \( x_1 = y \). The risk is \( R_2(p_c) = 2\alpha y - y \). Indeed, if also player 1 abandons the coalition \( p_c \) to play the selfish strategy \( x_1 = 0 \), player 2 would get \( y \) instead of \( 2\alpha y \). Consequently

\[
\tau_{1,\{2\}}(p_c) = \frac{y - \alpha y}{y - \alpha y + 2\alpha y - y} = \frac{1 - \alpha}{\alpha}.
\]

On the other hand, one has

\[ e_{1,\{2\}}(p_c) = \alpha y, \]

corresponding to player 2’s defection. Therefore,

\[ v_1(p_c) = v_2(p_c) = 2\alpha y \cdot \frac{2\alpha - 1}{\alpha} + \alpha y \cdot \frac{1 - \alpha}{\alpha} = (3\alpha - 1)y. \]

On the other hand, the selfish coalition \( p_s = (\{1\}, \{2\}) \) has value \( y \), corresponding to the equilibrium \((0, 0)\). Consequently,

\[ v(p_c) \leq v(p_s) \iff \alpha \leq \frac{2}{3}. \]

In other words: the exact cooperative equilibrium predicts tendency to cooperation as \( \alpha \) approaches 1. This is an interesting new prediction that we aim to verify or falsify by suitable experiments.

Example 5.6. We consider the finite version of Nash’s bargaining problem as in Example 4.4. It is well known that the unique reasonable solution is \((50, 50)\) and indeed a number of theories has been developed to select such a Nash equilibrium. For instance, in [Na50b], [Ka-Sm75], and [Ka77], the authors studied a set of additional axioms that guarantee that the unique solution of Nash bargaining problem is a 50-50 share. Other solutions have been recently proposed also in [Ha-Ro10] and [Ha-Pa12].

Now we show that also the cooperative equilibrium predicts a 50-50 share, if the two players have the same perception of gains.

Proposition 5.7. If the two players have the same perception of money, that is, \( f_1 = f_2 \), then the unique exact cooperative equilibrium is \((50, 50)\).
Proof. As we have already seen in Example 4.4, the cooperative partition $p_c$ has a unique acceptable profile of strategies, which is $(50, 50)$. Observe that $\text{Dev}(p) = \emptyset$ and therefore $\text{Ind}(G, p)$ is the game where both players can choose only the strategy 50. Consequently, we have

$$v_1(p_c) = v_2(p_c) = 50.$$ 

Now consider the selfish coalition profile $p_s = ((\{1\}, \{2\})$. This time the unique acceptable equilibria are

$$\text{Acc}_{\{1\}}(G_{p_s}) = (100, 0) \quad \text{Acc}_{\{2\}}(G_{p_s}) = (0, 100).$$

Observing that $\text{Dev}(p_s) = \emptyset$, we then obtain

$$v_1(p_s) = g_1(100, 100) = 0.$$

Analogously, we obtain $v_2(p_s) = g_2(100, 100) = 0$. Therefore the value of the cooperative coalition is larger than the value of the selfish coalition and, consequently, the set of exact cooperative equilibria of Nash bargaining problem coincides with the set of Nash equilibria of the induced game $\text{Ind}(G, p_c)$. Since this induced game contains only one profile of strategies, which is $(50, 50)$. This is then the unique exact cooperative equilibrium.

We now consider one more example where the predictions of the cooperative equilibrium are completely different from the predictions of any known solution concept and they are partially supported by experimental data.

Example 5.8. We consider the parametrized Prisoner’s dilemma as in Example 4.5. Observe that all known solution concepts predict either defection for sure or cooperation for sure. Nevertheless, the data collected for the parametrized Traveler’s dilemma suggest that human behavior in the parametrized Prisoner’s dilemma should depend on the parameter. We show that this is what happens using the cooperative equilibrium. Of course, it would be important to confirm such a prediction with a suitable experiment and make then a quantitative comparison between predictions and experimental data.

The cooperative coalition structure $p_c = ((\{1, 2\})$ gives rise to a one-player game whose Nash equilibrium is the cooperative profile $(C, C)$. The value of this coalition is, for both players,

$$v_1(p_c) = v_2(p_c) = (1 + \mu)\left(1 - \frac{1}{1 + \mu}\right) = \mu.$$

The selfish partition $p_s = ((\{1\}, \{2\})$ gives rise to the classical Nash equilibrium $(D, D)$. The value of $p_s$ is then, for both players,

$$v_1(p_s) = v_2(p_s) = 1.$$

Therefore, for $\mu > 1$, one has $v_1(p_c) = v_2(p_c) > v_1(p_s) = v_2(p_s)$ and the cooperative equilibrium predicts deviation towards cooperation and this deviation gets stronger and stronger when $\mu$ increases. On the other hand, when $\mu < 1$ one has $v_1(p_c) = v_2(p_c) < v_1(p_s) = v_2(p_s)$ and therefore the cooperative equilibrium predicts defection.

We mentioned in the Introduction that there are other solution concepts that have been proposed in the last few years and we have discussed why believe that Renou-Schlag-Halpern-Pass’s iterated regret minimization is the most promising of them: the others are either too rigid or inapplicable to one-shot games. Contrariwise, iterated regret minimization can explain deviations from Nash equilibria in several games. Nevertheless, as observed in [Ha-Pa12], it fails to predict human behavior for some other
games, such as the Prisoner’s dilemma and the Traveler’s dilemma with punishment. We have already computed the cooperative equilibrium for the Prisoner’s dilemma and we now make a parallelism between iterated regret minimization and cooperative equilibrium for the Traveler’s dilemma with punishment.

Example 5.9. Consider a variant of the Traveler’s dilemma that has been proposed in [Ha-Pa12], Section 6. Let us start from the Traveler’s dilemma in Example 4.6 where, this time, the strategy set is \{2, 3, ..., 100\} for both players and the bonus-penalty is \(b = 2\). Suppose that we modify this variant of the Traveler’s dilemma so as to allow a new action, called P (for punish), where both players get 2 if they both play P, but if one player plays P and the other plays an action other than P, then the player who plays P gets 2 and the other player gets \(−96\). In this case \((P, P)\) is a Nash equilibrium and it is also the solution in terms of regret minimisation. As observed in [Ha-Pa12], this is a quite unreasonable solution, since the intuition suggests that playing P should not be rational. In fact, one can easily check that, from our point of view, this game is absolutely the same as the original Traveler’s dilemma\(^{12}\) and therefore it has got the same cooperative equilibria.

We now give an example that we believe is relevant for the following reason: this is a game with multiple Nash equilibria and a unique evolutionary stable strategy, where the cooperative equilibrium is unique and it is very close to the evolutionary stable strategy. This is then one more example where, in case of multiple equilibria, the cooperative equilibrium is close to select the most natural one.

Example 5.10. Consider the following version of Hawk-Dove.

\[
\begin{array}{c|cc}
 & a_2 & b_2 \\
\hline
a_1 & 0, 0 & 7, 2 \\
b_1 & 2, 7 & 6, 6 \\
\end{array}
\]

This game has got three Nash equilibria: \((a_1, b_2)\), \((b_1, a_2)\), and \((\frac{1}{3}a_1 + \frac{2}{3}b_1, \frac{1}{3}a_2 + \frac{2}{3}b_2)\). However, the first two equilibria are quite unreasonable, because of their asymmetry. Indeed, in absence of uncorrelated asymmetry, the unique evolutionary stable strategy is \((\frac{1}{3}a_1 + \frac{2}{3}b_1, \frac{1}{3}a_2 + \frac{2}{3}b_2)\). We now show that the exact cooperative equilibrium is unique, it is quite close to the evolutionary stable strategy, and it is better than the evolutionary stable strategy, in the sense that if players play the cooperative equilibrium, then they would gain more than playing the evolutionary stable strategy.

The cooperative coalition structure \(p_c = (\{1, 2\})\) gives rise to a one-player game whose unique equilibrium is \((b_1, b_2)\), which must be acceptable (by uniqueness). Set \(i = 1\) (the case \(i = 2\) is the same, by symmetry). We have \(D_2(p_c) = 1\), corresponding to the deviation \(a_2\) from \(\sigma = (b_1, b_2)\), and \(R_2(p_c) = 6\), corresponding to the deviation \(a_1\) from \(\sigma\). Finally, we get \(e_{1,2}(p_c) = 0\) and \(e_{1,0}(p_c) = 6\). Therefore,

\[v_1(p_c) = v_2(p_c) = 6 \cdot \frac{6}{7} = \frac{36}{7}.
\]

Now we have to compute all profiles of mixed strategies \((\sigma_1, \sigma_2)\) such that

\[
\begin{align*}
g_1(\sigma_1, \sigma_2) &\geq \frac{36}{7} \\
g_2(\sigma_1, \sigma_2) &\geq \frac{36}{7}
\end{align*}
\]

\(^{12}\)Basiclly because strategies with very small payoff, such as P, do not enter in our computation of the value of the cooperative coalition.
To this end, set $\sigma_1 = \alpha_1 a_1 + (1 - \alpha_1) b_1$ and $\sigma_2 = \alpha_2 a_2 + (1 - \alpha_2) b_2$. From Equation 9 one gets

$$\begin{aligned}
\sigma_1 &= \alpha_1 - 4 \alpha_2 - 3 \alpha_1 \alpha_2 \geq -\frac{6}{7} \\
\sigma_2 &= -4 \alpha_1 - 3 \alpha_1 \alpha_2 \geq -\frac{6}{7}
\end{aligned}$$

(10)

These two hyperbolas intersect in a (convex and closed) region $R \subseteq [0, 1]^2$. Computing Nash equilibria in this region is quite straightforward. Indeed, assume that $\sigma_i = \sigma_i a_i + (1 - \alpha_i) b_i$ is a Nash equilibrium for player $i$, then the well known condition $g_1(\sigma_1, \sigma_2) = g_1(\sigma_1, \sigma_2)$, for all admissible $\sigma_1 = \alpha_1 a_1 + (1 - \alpha_1) b_1$, tells us that

$$\sigma_1 (1 - 3 \sigma_2) \geq \alpha_1 (1 - 3 \sigma_2),$$

for all admissible $\alpha_1$ (11)

Now, observe that the two hyperbolas in (10) intersects in the point

$$\left(\frac{-441 + \sqrt{427329}}{882}, \frac{-441 + \sqrt{427329}}{882}\right),$$

and the value $\alpha_+ = -441 + \sqrt{427329} \approx 0.24$ is also the maximal value admissible for $\alpha$ in the induced game $\text{Ind}(G, p_c)$. Observe that $\alpha_+ < \frac{1}{3}$ and therefore we can simplify $1 - 3 \sigma_2 in (11)$ and obtain the condition $\alpha_1 \leq \sigma_1$, for all $\alpha_1$ admissible; i.e., $\sigma_1 = \frac{-441 + \sqrt{427329}}{882}$. Analogously, we get the same value for $\sigma_2$.

Now let us analyse the selfish coalition structure $p_s = \{\{1\}, \{2\}\}$. This coalition gives rise to the classical game, which has three Nash equilibria, $(a_1, b_2)$, $(b_1, a_2)$, and $(\frac{1}{3} a_1 + \frac{2}{3} b_1, \frac{1}{3} a_2 + \frac{2}{3} b_2)$. The unique acceptable equilibrium for player 1 is $(a_1, b_2)$. Indeed, in the trivial coalition there is nothing to share and therefore, acceptable equilibria reduces to equilibria with largest payoff. Analogously, the unique acceptable equilibrium of player 2 is $(b_1, a_2)$. Therefore, if we compute the value of the coalition we obtain

$$v_1(p_s) = g_1(a_1, a_2) = 0 \quad \text{and} \quad v_2(p_s) = g_2(a_1, a_2) = 0.$$

Therefore, the value of the cooperative coalition is larger than the value of the selfish coalition and, consequently, the exact cooperative equilibrium predicts the solution

$$(\alpha_+ a_1 + (1 - \alpha_+) b_1, \alpha_+ a_2 + (1 - \alpha_+) b_2),$$

with $\alpha_+ = -441 + \sqrt{427329} \approx 0.24$.

As announced, this is slightly different from the evolutionary stable strategy and it is better than the evolutionary stable strategy in the sense that if both players play the cooperative equilibrium, then they would gain more than playing the evolutionary stable strategy. In one case they would get about $\frac{36}{7} > 5$ and in the other they would get $32/9$ which is even less that 4.

6. Towards cumulative prospect theory

In the previous section we have discussed a set of example where the cooperative equilibrium under expected utility theory predicts human behavior satisfactorily well. The aim of this section is to describe a set of example where such cooperative equilibrium fails to make satisfactorily precise predictions. All these examples have a natural explanation passing to cumulative prospect theory. This gives us one more reason to do this further step.

13In the sense that $\langle \sigma_1, \sigma_2 \rangle$ must be a solution of [10].
Example 6.1. Consider a version of the Prisoner’s dilemma experimented in [Co-DJ-Fo-Ro96], that is, the game described by the following gains

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>800, 800</td>
<td>0, 1000</td>
</tr>
<tr>
<td>D</td>
<td>1000, 0</td>
<td>350, 350</td>
</tr>
</tbody>
</table>

The cooperative coalition structure \( p_c = (\{1, 2\}) \) has the same value for both players, that is,

\[ v_1(p_c) = v_2(p_c) = 800 \left( 1 - \frac{200}{650} \right) \sim 554. \]

Now, to find the induced game \( \text{Ind}(G, p) \), we have to compute all profiles of mixed strategies \( \sigma = (\sigma_1, \sigma_2) \) such that \( g_1(\sigma) \geq 554 \) and \( g_2(\sigma) \geq 554 \). Setting \( \sigma_i = \alpha_i C + (1 - \alpha_i) D \), for \( i = 1, 2 \), one finds the conditions

\[
\begin{align*}
150\alpha_1\alpha_2 + 650\alpha_2 - 350\alpha_1 & \geq 204, \\
150\alpha_1\alpha_2 + 650\alpha_1 - 350\alpha_2 & \geq 204.
\end{align*}
\]

By symmetry, it is clear that the cooperative equilibrium must have the shape \( (\alpha C + (1 - \alpha)D, \alpha C + (1 - \alpha)D) \). So we can compute it just finding the smallest \( \alpha \) such that \( (\alpha, \alpha) \) is a solution of the system in (12), which is \( \alpha \sim 0.76 \). So we get that the unique Nash equilibrium of the induced game is \( (\alpha_+ C + (1 - \alpha_+)D, \alpha_+ C + (1 - \alpha_+)D) \).

On the other hand, the selfish coalition structure generates the classical game and therefore it has the unique equilibrium \( (D, D) \), whose value is 350. Therefore, the exact cooperative equilibrium predicts (approximatively) the strategy \( \alpha_+ C + (1 - \alpha_+)D \) for both players. Notice that in [Co-DJ-Fo-Ro96], Table 1, it has been reported that human players had been cooperating on the one-shot version of this game with a frequency of 0.22. Therefore, in this case, the cooperative equilibrium is able to predict qualitatively the tendency to cooperate, but the quantitative prediction is quite different.

This quantitative deviation has a natural explanation passing to cumulative prospect theory. Motivations and basic notions of cumulative prospect theory will be recalled in the next section. For now, let us just say that given a prospect \( (x, 1 - r; y, r) \), where \( x > y > 0 \) are the monetary outcomes, coming with probability \( 1 - r \) and \( r \), respectively, cumulative prospect theory postulates that a decision maker would evaluate the prospect \( p \) using a value function of the form

\[ V(x, 1 - r; y, r) = v(y) + \pi(1 - r)(v(x) - v(y)), \]

for suitable functions \( v \) and \( \pi \). We will recall in the next section the relevant literature about these two functions. For now, let us only say that the \( v(x) \) represents a sort of net value of the outcome \( x \); whereas \( \pi(a) \) is a weighting function having the characteristic of underweighting large probabilities. In this example we assume \( v(x) = x \) and we postpone to Section 3 the discussion about the naturality of this assumption. There are several attempts to find an explicit description of the function \( \pi \). For instance, the function proposed in [Tv-Ka92], fig. 3, has the property that \( \pi \left( \frac{2}{3} \right) \sim \frac{1}{2} \). To be honest, this estimation is not very precise, but the purpose of this example is just to show how the application of cumulative prospect theory can change the predictions in a quantitative way, leading to results that are closer to experimental data.

Coming back to our case, observe that evaluation of a coalition profile coincides with the evaluation of a certain prospect. Precisely, we have the prospect

\[ (e_{1,0}(p_c), \tau_{1,0}(p_c); e_{1,1}(p_c), \tau_{1,1}(p_c))(p_c)), \]
whose value, under expected utility theory, coincides with \( v_1(p_c) \). Using cumulative prospect theory, we would get the different value

\[
v_1^{\text{CPT}}(p_c) = \pi \left( \frac{400}{650} \right) \cdot 800.
\]

Using the approximation \( \frac{400}{650} \sim \frac{2}{3} \) (that is allowed by the simplification procedure described in [Ka-Tv79], p. 275) and then the approximation \( \pi \left( \frac{2}{3} \right) \sim \frac{1}{2} \), we would get \( v_1^{\text{CPT}}(p_c) \sim 400 \). Computing the Nash equilibrium of the induced game using this new value leads to quite different result. Indeed, the reader can easily show that this time the cooperative equilibrium would predict cooperation with probability about 0.27, that is much closer to the experimental data.

Observe that this application of prospect theory would not destroy the examples given in Section 5. Even more, the predictions are more precise. Indeed,

- In the Traveler’s dilemma with \( b = 2 \) and \( S_1 = S_2 = \{2, 3, \ldots, 100\} \), we had \( \tau_1,\{2\}(p_c) = \frac{1}{5} \) and
  \[
v(p_c) = 100 \cdot \frac{4}{5} + 96 \cdot \frac{1}{5}.
\]
  Consequently, overestimating \( \tau_1,\{2\}(p_c) \) we would get just a prediction which is closer to 96. This prediction is even more precise than the one we had.
- The same thing happen for \( b = 5 \).
- For \( b \geq 10 \) observe that \( \tau_1,\{2\}(p_c) \) is very close to \( \frac{1}{2} \) and therefore there is virtually no overestimation or underestimation, since the weighting function \( w \) is known to fix approximatively the probabilities near to \( \frac{1}{2} \).
- In the Bertrand competition with \( N = 2 \), one has \( \tau_1,\{2\}(p_c) = \frac{49}{99} \sim \frac{1}{2} \) and so, again, the prediction is basically the same.
- In the Bertrand competition with \( n = 4 \), one has
  \[
v(p_c) = 50 \cdot (1 - \tau_1,\{2,3,4\}(p_c)).
\]
  Now, we have \( \tau_1,\{2,3,4\} \sim \frac{7}{8} \) and therefore, in cumulative prospect theory, this probability should be overweighted. This implies that the value of \( p_c \) will gets larger than 6.5 and consequently the cooperative equilibrium would predict something even closer to the experimental data.
- In the Kahneman-Knetsch-Thaler’s experiment, the probability was \( \frac{1}{2} \) and therefore there are virtually no changes.
- In the other examples we did not do comparisons with experimental data. However, observe that the new predictions about the rate of cooperation in the parametrized Prisoner’s dilemma and in the public good game when \( \alpha \to 1 \) remain qualitatively, even though they can be quantitatively different.

We now consider other two games for which cumulative prospect theory can play an active and decisive role.

**Example 6.2.** Consider a parametrized generalized coordination game as follows.

\[
\begin{array}{c|cc}
  & a & b \\
\hline
  a & 1, 1 & 0, -k \\
  b & -k, 0 & 10, 10
\end{array}
\]
where \( k > 0 \). The unique exact cooperative equilibrium coincides with the Pareto optimal Nash equilibrium \((b, b)\) independently of \( k \). It is not clear if this is a reasonable prediction or not. Indeed, if \( k \) is very large, then the players may prefer not to risk and play the safe strategy \( a \). Observe, indeed, that the strategy \( a \) is risk dominant for \( k > 9 \) and that also iterated regret minimization coincides with the risk dominant profile of strategies. On the other hand, players should be clever enough to understand that there is no reason not to coordinate on \((b, b)\): no players have incentives to deviate. Of course, this dilemma may be solved only after seeing human behavior for this game: do really players change so discontinuously behavior when \( k \) passes from values slightly smaller than 9 to values slightly larger than 9? Do really players change behavior for large enough \( k \)? In absence of a suitable experiment, what we can say is that such a discontinuity could be explained in two different ways. The first one is by using a sort of quantal cooperative equilibrium under cumulative prospect theory. As we will remind in the next section, cumulative prospect theory takes into account risk aversion and, consequently, the use of cumulative prospect theory would explain why agents play the profile of strategies \((a, a)\): they simply do not consider playing \( b \) since this strategy is perceived to be too risky if \( k \) is large enough. This explanation might not seem very satisfactory, since it ultimately relies on the mistakes that players can make and this is why it requires a quantal version of cooperative equilibrium. Indeed, a player can delete the strategy \( b \) only if it gives positive probability to the event that the other agent plays the strategy \( a \). This can happen only in quantal cooperative equilibrium.

An explanation relying only in risk aversion could be that of modifying the definition of the value of a coalition. Indeed, we have assumed that a player \( j \) can abandon a coalition structure \( p \) only if \( j \) has a strategy which gives a larger gain to \( j \). Nevertheless, it is possible that another class of deviations are possible: deviation towards a strategy that gives a safe payoff. This alternative appears very fascinating but it requires a deeper analysis of the value of the coalition that can be motivated only if experiments will show such kinds of behavior.

**Example 6.3.** The following game has been proposed by J. Halpern in a private communication. Two players have the same strategy set \( \{a, b, c\} \) and the gains are described by the matrix

\[
\begin{array}{ccc}
  & a & b & c \\
 a & x, x & 0, 0 & 0, y \\
b & 0, 0 & x, x & 0, y \\
c & y, 0 & y, 0 & y, y \\
\end{array}
\]

where \( x > y > 0 \). In this case one finds \( v(p_c) = v(p_a) = 0 \) and consequently, the set of exact cooperative equilibrium is equal to the set of Nash equilibria. Nevertheless, in this case it is very likely that if \( y \) and \( x \) are very close and much larger than 0, then the two players would coordinate and play the safe strategy \( c \). Also this behavior would be predicted by the cooperative under cumulative equilibrium prospect theory: the strategies \( a \) and \( b \) are deleted a priori since perceived too risky with respect to the safe strategy. Observe that this example is apparently similar to the previous one but it is profoundly different: it does not require the quantal cooperative equilibrium. Therefore, the explanation of the coordination on the safe strategy \( c \) relies only in risk aversion.
A SOLUTION CONCEPT FOR GAMES WITH ALTRUISM AND COOPERATION

7. A BRIEF INTRODUCTION TO CUMULATIVE PROSPECT THEORY

The examples described in the previous sections give one more motivation to abandon expected utility theory and use cumulative prospect theory. Before starting the description of the cooperative equilibrium under cumulative prospect theory, we take this short section to give a short introduction to this theory.

By definition, a prospect \( p = (x_{-m}, p_{-m}; \ldots; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; \ldots; x_n, p_n) \) yields outcomes \( x_{-m} < \ldots < x_{-1} < x_0 = 0 < x_1 < \ldots < x_n \) with probabilities \( p_i > 0 \), for \( i \neq 0 \), and \( p_0 \geq 0 \) that sum up to 1.

Expected utility theory was founded by Morgenstern and von Neumann in \[Mo-vN47\] to predict the behavior of a decision maker that must choose a prospect among some. Under certain axioms (see, for instance, \[Fi82\]) Morgenstern and von Neumann proved that a decision maker would evaluate each prospect \( p \) using the value

\[
V(p) = \sum_{i=-m}^{n} p_i u(x_i),
\]

(13)

where \( u(x_i) \) is the utility of the outcome \( x_i \), and then she would choose the prospect(s) maximizing \( V(p) \).

It has been first realized by M. Allais in \[Al53\] that a human decision maker does not really follow the axioms of expected utility theory and, in particular, she evaluates a prospect using an evaluation procedure different from the one in (13). A first attempt to replace expected utility theory with a theory founded on different axioms and able to explain deviations from rationality was done in \[Ka-Tv79\] where Kahneman and Tversky founded the so-called prospect theory. This novel theory encountered two problems. First, it did not always satisfy stochastic dominance, an assumption that many theorists were reluctant to give up. Second, it was not readily extendable to prospects with a large number of outcomes. Both problems could be solved by the rank-dependent or cumulative functional, first proposed by Quiggin \[Qu82\] for decision under risk and by Schmeidler \[Sc89\] for decision under uncertainty. Finally, Kahneman and Tversky were able to incorporate the ideas presented in \[Qu82\] and \[Sc89\] and develop their cumulative prospect theory in \[Tv-Ka92\]. Prospect theory and cumulative prospect theory have been successfully applied to explain a large number of phenomena that expected utility theory was not able to explain, as the disposition effect \[Sh-St85\], asymmetric price elasticity \[Pu92\], \[Ha-Jo-Fa93\], tax evasion \[Dh-No07\], as well as many problems in international relations \[Le92\], finance \[Th05\], political science \[Le03\], among many others.

The basic principles of cumulative prospect theory are the first four of the following list. The fifth one has been recently proposed by al-Nowaihi and S. Dhami, who provided an extensive evidence in \[No-Dh11\], Section 2.

(P1) Decision makers weight probabilities in a non linear manner. In particular, the evidence suggests that decision makers overweight low probabilities and underweight high probabilities.

---

14 Prospect theory and cumulative prospect theory have been originally developed for monetary outcomes (see \[Ka-Tv79\], p.274, l.4), giving us one more motivation to abandon utility functions and work with gain functions. Kahneman and Tversky’s choice to work with monetary outcomes is probably due to the second principle of their theory, as it will be recalled little later.

15 The two papers in prospect theory and cumulative prospect theory have more than 30000 citations.
(P2) Decision makers think in terms of gains and losses rather than in terms of their net assets\(^{16}\).

(P3) Decision makers tend to be risk-averse with respect to gains and risk-acceptance with respect to losses\(^{17}\).

(P4) *Losses loom larger than gains*; namely, the aggravation that one experiences in losing a sum of money appears greater than the pleasure associated with gaining the same amount of money.

(P5) Decision makers ignore events of extremely low probability and treat extremely high probabilities as certain.

The consequence of these principles is that decision makers evaluate a prospect \(p\) using a value function

\[
V(p) = \sum_{j=-m}^{n} \pi_j v(x_j)
\]

that is completely different from the one in \((13)\). To understand the explicit shape of the functions \(v\) and \(\pi\) is probably the most important problem in cumulative prospect theory. About the function \(v\), it has been originally proposed in \(Tv-Ka92\) to use the function

\[
v(x) = \begin{cases} 
  x^\alpha, & \text{if } x \geq 0; \\
  -\lambda(-x)^\beta, & \text{if } x < 0.
\end{cases}
\]

where experiments done in \(Tv-Ka92\) gave the estimations \(\alpha \sim \beta \sim 0.88\) and \(\lambda \sim 2.25\). About the function \(\pi\), the situation is much more intrigued: cumulative prospect theory postulates the existence of a strictly increasing surjective function \(w: [0,1] \rightarrow [0,1]\) such that

\[
\pi_{-m} = w(p_{-m}) \\
\pi_{-m+1} = w(p_{-m} + p_{-m+1}) - w(p_{-m}) \\
\vdots \\
\pi_j = w\left(\sum_{i=-m}^{j} p_i\right) - w\left(\sum_{i=-m}^{j-1} p_i\right) \quad j < 0 \\
\pi_0 = 0 \\
\pi_j = w\left(\sum_{i=j}^{n} p_i\right) - w\left(\sum_{i=j+1}^{n} p_i\right) \quad j > 0 \\
\vdots \\
\pi_{n-1} = w(p_{n-1} + p_n) - w(p_n) \\
\pi_n = w(p_n)
\]

\(^{16}\)This principle is probably the one which forced Kahneman and Tversky to work with monetary outcomes and force us to work with gain functions.

\(^{17}\)As a consequence, risk aversion is already taken into account and this is why we did not need to consider it explicitly in the definition of a game in explicit form.
A first proposal of such a function \( w \) was made by Tversky and Kahnman themselves in [Tv-Ka92] and it is

\[
w(p) = \frac{p^\gamma}{(p^\gamma + (1 - p)^\gamma)^\frac{1}{\gamma}}
\]

where \( \gamma \) has been estimated to belong to the interval \([\frac{1}{2}, 1]\) in [Ri-Wa06]. Other functions \( w \) have been proposed in [Ka79], [Go-Ei87], [R87], [Cu-Sa89], [La-Ba-Wi92], [Lu-Me-Ch93], [He-Or94], [Pr98], and [Sa-Se98]. Nevertheless, all these functions turn out not to meet the principle (P5). To fill this gap, al-Nowaihi and Dhami [No-Dh11] have modified Prelec’s weighting function in [Pr98] and proposed the so-called composite Prelec weighting function, that looks like

\[
w(p) =
\begin{cases}
0, & \text{if } p = 0; \\
\exp(-\beta_0(-\ln(p))^\alpha_0), & \text{if } 0 < p \leq \overline{p}; \\
\exp(-\beta(-\ln(p))^\alpha), & \text{if } p < \overline{p} \leq \overline{p}_1; \\
\exp(-\beta_1(-\ln(p))^\alpha_1), & \text{if } \overline{p} < p \leq 1.
\end{cases}
\]

where \( 0 < \alpha < 1, \beta > 0, \alpha_0 > 1, \beta_0 > 0, \alpha_1 > 1, \beta_1 > 0, \beta_0 < 1/\beta^{\frac{\alpha_0-1}{\alpha_0}}, \beta_1 > 1/\beta^{\frac{\alpha_1-1}{\alpha_1}} \), \( p \) is close enough to 0 and \( \overline{p} \) is close enough to 1. For instance, [No-Dh11], Fig. 6.1, is obtained with

\[
\overline{p} = \exp\left(-\left(\frac{\alpha}{\alpha_0}\right)^{\frac{1}{\alpha_0-\alpha}}\right), \quad \overline{p}_1 = \exp\left(-\left(\frac{\alpha}{\alpha_1}\right)^{\frac{1}{\alpha_1-\alpha}}\right).
\]

It is not our purpose to give too many details about the enormous literature devoted to understanding the evaluation procedure in cumulative prospect theory. Our purposes were indeed to give a brief introduction to the theory and stress how this theory implies the necessity to work with gain functions instead of utility functions. So we now pass to the description of the cooperative equilibrium for finite games in explicit form under cumulative prospect theory and taking into account altruism.

**8. Iterated Deletion: The Set of Playable Strategies**

The cooperative equilibrium under cumulative prospect theory and taking into account altruism will be defined through two steps. In the first step we use the altruism parameters \( a_{ij} \) to eliminate the strategies that are not good for the collectivity. The second step is the prospect theoretical analogue of the procedure described in Section 4 applied to the subgame obtained after eliminating the strategies in the first step.

In this section we describe the first step of the construction, that we call *iterated deletion*. As it is well known, iterated deletion of strategies is a procedure which is common to most solution concepts (in Nash theory, one deletes dominated strategies; in iterated regret minimization theory, one deletes strategies which do not minimize regret; in Bernheim’s and Pearce’s rationality theory ([Be84] and [Pe84]), one deletes strategies that are not *justifiable* [Os-Ru94]). However, the use of altruism parameters to delete strategies seems new in the literature. This iterated deletion of strategies is based on a new notion of domination between strategies, that we call super-domination, which is motivated by the fact that human players do not eliminate weakly or strongly dominated strategies (as shown by the failure of the classical theory to predict human behavior in the Prisoner’s and Traveler’s Dilemmas).
Each step of our iterated deletion of strategies is made by two sub-reductions. The first sub-reduction is based on the following principle:

(CS) If \( s_i \in S_i \) is a strategy for which there is another strategy \( s'_i \in S_i \) which gives a certain larger gain (or a certain smaller loss) to player \( i \) and does not harm too much the other players, then player \( i \) will prefer the strategy \( s'_i \) and will never ever play the strategy \( s_i \).

Thus, this principle states that every player is selfish unless the society gets a big damage. Implicit in this principle there is a new notion of domination between strategies.

**Definition 8.1.** Let \( s_i, s'_i \in S_i \). We say that \( s_i \) is super-dominated by \( s'_i \) and we write \( s_i <_i s'_i \), if

1. for all \( s_{-i}, s'_{-i} \in S_{-i} \), one has \( g_i (s_i, s_{-i}) \leq g_i (s'_i, s'_{-i}) \),
2. there are \( s_{-i}, s'_{-i} \in S_{-i} \) such that \( g_i (s_i, s_{-i}) < g_i (s'_i, s'_{-i}) \).

Observe that super-domination is much stronger than the classical notion of weak domination. This makes sense since it has been observed that in many situations, as in the Traveler’s dilemma, players do not eliminate weakly dominated strategies, while it is clear that a purely selfish player would delete a super-dominated strategy. On the other hand, there is no direct relation between super-domination and strong-domination, as shown by the following examples.

**Example 8.2.** Consider the following version of the Prisoner’s dilemma

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

The strategy D strongly dominates U and the strategy R strongly dominates the strategy L. Nevertheless, there are no super-dominated strategies, since \( g_1(D, R) < g_1(U, L) \) and \( g_2(D, R) < g_2(U, L) \).

**Example 8.3.** Consider the two-person zero-sum game

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,0</td>
<td>10, -10</td>
</tr>
<tr>
<td>D</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

In this case L super-dominates R, but R is not strongly dominated by L, since \( g_2(D, R) = g_2(D, L) \).

We will see in Example 8.12 that the notion of super-domination between strategies can be interesting in itself, since it allows to explain some phenomena that are not easy to capture making use of weakly and strongly dominated strategies.

Before coming back to the theory, we need to fix some terminology. Fix \( \sigma_i \in \mathcal{P}(S_i) \), the fiber game defined by \( \sigma_i \) is the \((N-1)\)-player game \( \mathcal{G}_{\sigma_i} \), obtained by \( \mathcal{G} \) assuming that player \( i \) plays the strategy \( \sigma_i \) surely. Formally, \( \mathcal{G}_{\sigma_i} = \mathcal{G}(P \setminus \{i\}, S_{-i}, g_{\sigma_i}, a, f) \), where \( g_{\sigma_i} \) is the \((N-1)\)-dimensional vector whose components are the functions \( g_j(\sigma_i, \cdot) \), for \( j \in P \setminus \{i\} \). Using a trick which is conceptually similar to the one used in [Ha-Ro10], we define the cooperative equilibrium by induction on the number of players.

**Definition 8.4.** The cooperative equilibria of a one-player game are all probability measures supported on the set of pure strategies that maximize the gain function.
Now we suppose that we have already defined the cooperative equilibrium for all
\((N - 1)\)-player games and we define the cooperative equilibrium for all \(N\)-player games. We denote by \(\text{Coop}(\mathcal{G})\) the set of cooperative equilibria of a game \(\mathcal{G}\).

Now, fix \(i \in P\) and let \(s, t \in S_i\), with \(s <_i t\). If player \(i\) is believed to play the strategy \(t\), the other players would answer playing an \textit{equilibrium} of the fiber game \(\mathcal{G}_t\). Since the fiber game has \(N - 1\) players, we may use the inductive hypothesis. We define the \textit{set of losers} to be

\[
L_i(s, t) := \left\{ j \in P : g_j \left( s, \sigma_{-i}^{(s)} \right) > g_j \left( t, \sigma_{-i}^{(t)} \right), \text{ for all } \sigma_{-i}^{(t)} \in \text{Coop}(\mathcal{G}_t), \sigma_{-i}^{(s)} \in \text{Coop}(\mathcal{G}_s) \right\}.
\]

In words, \(L_i(s, t)\) is the set of players that have a certain disadvantage when player \(i\) decides to play the strategy \(t\) instead of her worse strategy \(s\). Now player \(i\) computes the lowest proportion among the gain that she can get and the loss that \(j \in L_i(s, t)\) can get. Therefore, for all \(j \in L_i(s, t)\), we define the number

\[
D_{i,j}(s, t) := \inf \left\{ \frac{g_i \left( t, \sigma_{-i}^{(t)} \right) - g_i \left( s, \sigma_{-i}^{(s)} \right)}{g_j \left( s, \sigma_{-i}^{(s)} \right) - g_j \left( t, \sigma_{-i}^{(t)} \right)} : \sigma_{-i}^{(t)} \in \text{Coop}(\mathcal{G}_t), \sigma_{-i}^{(s)} \in \text{Coop}(\mathcal{G}_s) \right\}
\]

If \(D_{i,j}(s, t)\) is large, then a big gain for player \(i\) corresponds to a small disadvantage for the players that are penalized by the choice of \(t\); therefore, player \(i\) will not choose \(s\) and will choose \(t\). This point of view puts in evidence that the principle (CS) describes a sort of \textit{controlled selfishness}: player \(i\) is selfish unless she harms the other players too much. Therefore, recalling the altruism parameter \(a_{ij}\) represents the units of the good \(g\) that players \(i\) is available to renounce in order to favor player \(j\) of one unit of the good, we give the following

\textbf{Definition 8.5.} A strategy \(s \in S_i\) is \textit{unplayable of the first type} for player \(i\) if there is another strategy \(t \in S_i\) such that

- \(s <_i t\)
- for all \(j \in L_i(s, t)\), one has \(D_{i,j}(s, t) \geq a_{ij}\).

In this case we write \(s <_1^i t\).

\textbf{Example 8.6.} Consider the game with gain matrix

\[
\begin{array}{ccc}
L & R \\
U & 1,1 & 1,1 \\
D & 1,1 & 2,1 \\
\end{array}
\]

Observe that \(U <_1 D\). Moreover, \(L_1(U, D) = \emptyset\) and therefore the second condition in Definition 8.5 is true for trivial reasons. Consequently, the strategy \(U\) is unplayable for the first player. This happens, roughly speaking, because the vertical player, playing \(D\), can have a gain without damaging the horizontal player.

\textbf{Example 8.7.} A little less trivial example is given by the game represented by the following gain matrix

\[
\begin{array}{ccc}
L & R \\
U & 0,0 & 0,0 \\
D & 1,-1 & 1,-1 \\
\end{array}
\]

Assume \(a_{12} \leq 1\). Of course, \(U <_1 D\). Now, observe that \(D_{1,2}(U, D) = 1 \geq a_{12}\), thus the strategy \(U\) is unplayable for the vertical player. Roughly speaking, this happens because the vertical player, playing \(D\), will get a certain gain giving a damage to the horizontal player that is small compared to her gain.
Coming back to the theory, we would like to delete unplayable strategies of the first type. To this end, we need to prove a simple lemma. Given \( s_i \in S_i \), let Maj\(^I\)\((s_i) = \{s'_i \in S_i : s_i <^I s'_i\}\).

**Lemma 8.8.** For all \( i \in P \), there exists \( s_i \in S_i \) such that Maj\(^I\)\((s_i) = \emptyset\).

**Proof.** By contradiction, let Maj\(^I\)\((s_i) \neq \emptyset\), for all \( s_i \in S_i \). Fix \( s_i^{(1)} \in S_i \). An iteration of the property Maj\(^I\) \neq \emptyset allows to construct a chain

\[
s_i^{(1)} <^I s_i^{(2)} <^I \ldots <^I s_i^{(n)}
\]

By finiteness of the set \( S_i \), we may assume that at some point we get \( s_i^{(n)} = s_i^{(1)} \), with \( s_i^{(n-1)} \neq s_i^{(1)} \). Observe that the relation \(<^I_i\) might not be transitive, but the underlying relation \(<_i\) is transitive. Therefore, we have gotten

\[
s_i^{(1)} <_i s_i^{(n-1)} \quad \text{and} \quad s_i^{(n-1)} <_i s_i^{(1)}
\]

that contradict each other. \( \Box \)

Let UnPl\(_i\)\((\mathcal{G})\) be the set of player \( i \)'s unplayable strategies of the first type and denote by Pl\(_i\)\((\mathcal{G}) := S_i \setminus \text{UnPl}_i\)\((\mathcal{G})\), that is well defined and non-empty by Lemma 8.8. The notation Pl\(_i\)\((\mathcal{G})\) stands for the cartesian product of all the Pl\(_j\)\((\mathcal{G})\)'s but \( \text{Pl}_i\)\((\mathcal{G})\).

Now we start the description of the second sub-restriction, that will be done through the definition of unplayable strategies of the second type. The *principle* underlying this second restriction is somehow the dual principle of the one underlying the previous restriction:

(PA) If \( s \in S_i \) is a strategy for which there is another strategy \( t \in S_i \) such that player \( i \) has a little disadvantage, but the other players have a big advantage, then player \( i \) will prefer the strategy \( t \) in order to help the society.

As said earlier, the principle (CS) is a sort of controlled selfishness, whereas the principle (PA) sounds more like *pure altruism*. We can formalize it in a similar way as we formalized (CS). Indeed, we can use the number \( D_{ij}(s, t) \) in a dual way: if it is very small, then a very little lost of player \( i \) corresponds to a huge gain of player \( j \). Therefore, we propose the following

**Definition 8.9.** A strategy \( t \in \text{Pl}_i\)\((\mathcal{G})\) is called *unplayable of the second type* for player \( i \) if there is another strategy \( s \in \text{Pl}_i\)\((\mathcal{G})\) such that

1. \( s <_i t \),
2. There exists \( j \in P \setminus \{i\} \) such that \( D_{ij}(s, t) \leq a_{ij} \).

**Example 8.10.** Consider the dictator game. A proposer offers a division of 10 dollars, which the responder has to accept. The standard perfect equilibrium analysis of this games is that the proposer should keep all the money, since the responder has no say. Nevertheless, in experiments has been reported that most *proposers* offer a certain amount of money to the responder (see, for instance, [Ro-Ho-Sh-Se94]). Bolton and Ockenfels explained this anomalous behavior using equity in their celebrated paper [Bo-Oc00]. We can explain it using iterated deletion of strategies using altruism. Let us model the set of strategies of the proposer, for simplicity, by \( S = \{0, 1, \ldots, 10\} \). It is clear that there is a chain of super-dominated strategies \( 0 <_\text{prop} 1 <_\text{prop} 2 <_\text{prop} \ldots <_\text{prop} 10 \) and one can easily show that every strategy \( s \) with \( s < a_{\text{prop,resp}} \) is unplayable of the
second type. Therefore, cooperative equilibrium theory predicts that the proposer offers a fairer division because of altruism: larger is $a_{\text{prop,resp}}$, larger is the offer.

Let $\text{UnPl}^{(2)}(G)_i$ be the set of player $i$’s unplayable strategies of the second type and denote by $\text{Pl}^{(2)}(G)_i := \text{Pl}^{(1)}(G)_i \setminus \text{UnPl}^{(2)}(G)_i$. This set is well defined and non-empty thanks to the obvious analogue of Lemma 8.8.

Now, we start an iteration of this procedure: we consider the subgame $G_2$ of $G$ defined by the strategy sets $\text{Pl}^{(2)}(G)_i$ and we reduce again these strategy sets computing the unplayable strategies of the two types; in this way, we get other sets of playable strategies $\text{Pl}^{(2)}(G_2)_i$; and we start again the procedure. By finiteness of the strategy sets $S_i$, this iteration stabilizes, that is, at some step $k$, one has $\text{Pl}^{(2)}(G_k)_i = \text{Pl}^{(2)}(G_{k+1})_i$. We set $\text{Pl}_i := \text{Pl}^{(2)}(G_k)_i$.

**Definition 8.11.** The set $\text{Pl}_i$ is called set of playable strategies of player $i$.

Before starting the second step of the construction, that is, the prospect theoretical analogue of Section 4, we give more details about the game introduced in Example 8.3. Indeed, this game seems interesting from several viewpoints. First, it is an example where the procedure of elimination of unplayable strategies stabilizes after more than one step. Then, it is one more example where iterated regret minimization theory fails to predict the intuitively right behavior, whereas the cooperative equilibrium does apparently the right job. Finally, it is an example where super-dominated strategies turn out to be very helpful to modify iterated regret minimization theory allowing prior beliefs and consequently obtaining the right prediction also under iterated regret minimization theory.

**Example 8.12.** Consider the same two-person zero-sum game as in Example 8.3, that is, the game with gain matrix

\[
\begin{array}{cccc}
& L & R \\
U & 0, 0 & 10, -10 \\
D & 1, -1 & 1, -1 \\
\end{array}
\]

Assume$^{19}$ that $a_{12} \leq 1$. Observe that $L$ super-dominates $R$ and that $L_2(R, L) = \emptyset$. Consequently, $R$ is unplayable of the first type. On the other hand, in this first step $U$ and $D$ are not ordered and therefore, the first step of the reduction leads to the subgame $G_2$ where the vertical player still has both strategies $U$ and $D$ available, whereas the horizontal player has only the strategy $L$. Therefore, one more application of deletion of unplayable strategies (using the assumption $a_{12} \leq 1$) leads to the trivial game where the vertical player has only the strategy $D$ and the horizontal player has only the strategy $L$. Therefore, $(D, L)$ is the unique cooperative equilibrium of this game. Observe that this is also a Nash equilibrium. The other Nash equilibrium is $(D, \frac{9}{10}L + \frac{1}{10}R)$, as one can easily check, which is quite unreasonable, since there is no reason why the horizontal player should play $R$: playing $L$ she will get certainly at least the same as playing $R$. Therefore, the cooperative equilibrium coincides with the most reasonable Nash equilibrium.

$^{19}$Observe, en passant, that this is a very reasonable assumption for players that meet for the first time in a zero sum game and having the same perception of gains: why should player $i$ renounce to more than one unit of her gain to favourite player $j$ of one unit of his gain, if $i$ meets $j$ for the first time, they have the same perception of money, and the game is a zero-sum game? In general, this will not happen.
On the other hand, a direct application of the iterated regret minimization procedure predicts that the vertical player actually plays U surely. This is also quite unreasonable, because playing U makes sense only if the column-player plays R. This cannot happen, above all if the column-player understands that the row-player can play U. As suggested by Halpern in a private communication, one can fix this problem allowing prior beliefs, in a conceptually similar way as in [Ha-Pa12], Section 3.5: first one eliminates weakly dominated strategies, then applies iterated regret minimization. Nevertheless, this procedure is questionable on one point: it is not clear why one should eliminate weakly dominated strategies in this context and not in the Traveler’s dilemma. One can fix this problem using super-domination. If one eliminates super-dominated strategies in the game under consideration before applying iterated regret minimization, one finds the right solution (D,L), coherently with the classical theory and the cooperative equilibrium. Moreover this is perfectly coherent with the other examples discussed in [Ha-Pa12] and in particular with the Traveler’s dilemma: the Traveler’s dilemma has many weakly dominated strategies, but none of them is super-dominated.

9. The cooperative equilibrium under cumulative prospect theory

In this section we finally define the cooperative equilibrium for games in explicit form \( G = \mathcal{G}(P, S, g, a, f) \) in complete generality. In the previous section we have restricted the sets of pure strategies and we have defined the sets of playable strategies \( P_i \). We denote by \( \text{Red}(\mathcal{G}) \) this reduced game, that is, the subgame of \( \mathcal{G} \) defined by the strategy subsets \( P_i \). The cooperative equilibrium for \( \mathcal{G} \) (under prospect theory and taking into account altruism) will be obtained by applying the construction described in Section 4 to the reduced game \( \text{Red}(\mathcal{G}) \) and making use of cumulative prospect theory. To this end, notice that the construction presented in Section 4 depends on expected utility theory only on two points:

(1) We have used expected utility theory to compute the value of the prospect

\[
(e_i, J(p), \tau_i, J(p))
\]

indexed by \( J \subseteq P \setminus \{i\} \). Using cumulative prospect theory, the value that we called \( v_i(p) \) should be replaced by its prospect theoretical analogue

\[
v_i^{\text{CPT}}(p) = \sum_{J \subseteq P \setminus \{i\}} v(e_i, J(p))^{\tau_i, J(p)}.
\]  

But now also the definition of the induced game should be modified: indeed we should allow only the profiles of strategies \( \sigma \) such that \( v(g_i(\sigma)) \geq v_i^{\text{CPT}}(p) \).

Observe that the two applications of the function \( v \), the first in the computation of \( v_i^{\text{CPT}} \) and the second in the definition of the induced game, are somehow inverse. Indeed, if \( v \) were linear and increasing, the induced game would have been the same as the one obtained by setting \( v(x) = x \). Now, we know from cumulative prospect theory that \( v \) is strictly increasing. Approximating it by a linear function we can simply a lot the definition and set \( v(x) = x \). This explains why the examples in Section 6 fit the experimental data very well: they have been conducted with relatively small monetary outcomes and there were

\[\text{[12]}\] If one eliminates weakly dominates strategies in the Traveler’s dilemma before applying iterated regret minimization, one obtains the Nash equilibrium.
no possible losses. Of course, it is predictable that in case of possible large gains and/or losses, this approximation will create problems.

(2) The definition of the value of a coalition and then the definition of the cooperative equilibrium rely in the computation of Nash equilibria of the games $G_p$ and $\text{Ind}(G, p)$. The computation of Nash equilibria uses expected utility theory, precisely in the definition of the mixed extension of gain functions. Unfortunately, the natural translation of Nash equilibrium in the language of cumulative prospect theory leads to define an object that might not exist (see [Cr90] and, more generally, [Fi-Pa10]). To avoid this problem we consider a solution concept which is a bit more general than Nash equilibrium, the so-called equilibrium in beliefs, introduced by Crawford in [Cr90]. Crawford’s equilibria in beliefs have the good property to exist in our context, contain all Nash equilibria, and reduce to Nash equilibria in many cases. The remainder of the section is devoted to this.

Before recalling the definition of an equilibrium in beliefs, we need to do a preliminary step, that is writing the mixed extension of the gain functions in the language of cumulative prospect theory. Since notation will get complicated very soon, we start by an example.

Example 9.1. Consider the (already reduced) game with gain matrix:

\[
\begin{array}{cc}
C & D \\
C & 2, 2 & 0, 3 \\
D & 3, 0 & 1, 1 \\
\end{array}
\]

Assume that the vertical player (player 1) plays the mixed strategy $\sigma_1 = \frac{1}{8} C + \frac{7}{8} D$ and player 2 plays the mixed strategy $\sigma_2 = \frac{1}{4} C + \frac{3}{4} D$. Under expected utility theory, we would have

\[g_1(\sigma_1, \sigma_2) = \sum_{x \in S_1} \sum_{y \in S_2} g_1(x, y) \sigma_1(x) \sigma_2(y) \]

Let us compute step by step this number to put in evidence where and how expected utility theory must be replaced by cumulative prospect theory. Fix $\sigma_2$ as before and observe that we have a finite family of prospects, one for each pure strategy of the first player. In this example, they are:

\[p(C, \sigma_2) = \left( \frac{2}{4}, \frac{1}{4}; 0, \frac{3}{4} \right) \quad \text{and} \quad p(D, \sigma_2) = \left( \frac{3}{4}, \frac{1}{4}; 1, \frac{3}{4} \right) \]

Now, under expected utility theory (and this is the first point where expected utility theory is used), one computes the values of the two prospects, obtaining, in this particular example, the values

\[V_1(C, \sigma_2) = 2 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = \frac{1}{2} \quad \text{and} \quad V_1(D, \sigma_2) = 3 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{2} \]

Of course, these numbers are equal to the ones that are usually denoted by $g_1(C, \sigma_2)$ and $g_2(D, \sigma_2)$, respectively. Now, to compute the value usually denoted by $g_1(\sigma_1, \sigma_2)$, one first constructs one more prospect using the measure $\sigma_1$, that is

\[p(\sigma_1, \sigma_2) = \left( \frac{1}{2}, \frac{1}{8}; \frac{3}{2}, \frac{7}{8} \right) \]

and finally, again under expected utility theory, one computes the value of this prospect, obtaining the well known value $g_1(\sigma_1, \sigma_2)$. 
We want to replace the classical values $g_i(\sigma_i, \sigma_{-i})$ with new values $V_i(\sigma_i, \sigma_{-i})$, obtained replacing expected utility theory with cumulative prospect theory. From the example, it is clear that, to compute $V_i(\sigma_i, \sigma_{-i})$ in cumulative prospect theory, we only need to compute first $V_i(s_i, \sigma_{-i})$, for all $s_i \in P_i$, using cumulative prospect theory on the prospects $p^{(s_i, \sigma_{-i})}$, and then compute $V_i(\sigma_i, \sigma_{2})$ using cumulative prospect theory on the prospect $p^{(\sigma_i, \sigma_{-i})}$. To make this idea formal, recall that in cumulative prospect theory the outcomes of a prospect are supposed to be ordered in increasing way. It is then useful to associate to each prospect $p = (x_1, p_1; \ldots; x_n, p_n)$, with distinct outcomes $x_i \in \mathbb{R}$, a permutation $\rho(p)$ that is just the permutation of the $x_i$'s such that $\rho(p)(x_i) < \rho(p)(x_{i+1})$, for all $i$. Now for all $(s_i, s_{-i}) \in P_i \times P_{-i}$, we define

$$A_i^{(s_i, s_{-i})} := \{ s'_i \in P_{-i} : g_i(s_i, s_{-i}) = g_i(s_i, s'_i) \}.$$  

For any fixed $s_i$, the sets $A_i$'s form a partition of $P_{-i}$. Choose a transversal $T_{s_i}$ for this partition, that is, $T_{s_i}$ is a subset of $P_{-i}$ constructed picking exactly one point for each set $A_i$. Now fix $(\sigma_i, \sigma_{-i}) \in \mathcal{P}(P_i) \times \mathcal{P}(P_{-i})$ and define the prospect

$$p_{(\sigma_i, \sigma_{-i})}^{(s_i, s_{-i})} = \left( g_i(s_i, s_{-i}), \sigma_{-i} \left( A_i^{(s_i, s_{-i})} \right) \right),$$

where $s_{-i}$ runs over the transversal $T_{s_i}$. Of course, this prospect does not depend on the particular transversal we fixed. Now, the outcomes of this prospect might not be ordered in increasing way. Therefore, before applying cumulative prospect theory to compute $V_i(s_i, p_{(\sigma_i, \sigma_{-i})})$ we must apply the permutation $\rho(p_{(\sigma_i, \sigma_{-i})})$. Consequently, with the notation as in Section 7, we obtain

$$V_i(s_i, p_{(\sigma_i, \sigma_{-i})}) = \sum_{s_{-i} \in T_{s_i}} \pi_{s_{-i}} v \left( \rho \left( p_{(\sigma_i, \sigma_{-i})} \right) \right) \left( g_i(s_i, s_{-i}) \right).$$

To construct the second prospect $p_{(\sigma_i, \sigma_{-i})}$, we follow an analogous procedure. Let

$$B_i^{(s_i, \sigma_{-i})} = \{ s'_i \in P_i : V_i(s_i, \sigma_{-i}) = V_i(s_i', \sigma_{-i}) \}.$$  

The $B_i$'s form a partition of $P_i$. Let $T_{\sigma_{-i}}$ be a transversal for this partition. We define the prospect

$$p_{(\sigma_i, \sigma_{-i})} = \left( V_i(s_i, \sigma_{-i}), \sigma_{i} \left( B_i^{(s_i, \sigma_{-i})} \right) \right),$$

where $s_i$ runs over $T_{\sigma_{-i}}$. Therefore we obtain

$$V_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in T_{\sigma_{-i}}} \pi_{s_i} v \left( \rho \left( p_{(\sigma_i, \sigma_{-i})} \right) \right) \left( \sum_{s_{-i} \in T_{s_i}} \pi_{s_{-i}} v \left( \rho \left( p_{(\sigma_i, \sigma_{-i})} \right) \right) \left( g_i(s_i, s_{-i}) \right) \right).$$

One is now tempted to define a Nash equilibrium of a game under cumulative prospect theory as a profile $(\sigma_1, \ldots, \sigma_N)$ of mixed strategies such that for all $i \in P$ and for all $\sigma'_i \in \mathcal{P}(S_i)$ one has $V_i(\sigma_i, \sigma_{-i}) \geq V_i(\sigma'_i, \sigma_{-i})$. As mentioned before, unfortunately, there are games without Nash equilibria in this sense. To avoid this problem, we use Crawford’s trick to extend the set of Nash equilibria including the so-called equilibria in beliefs. To do that, first we recall the following classical definition.

**Definition 9.2.** Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex set and let $\phi : \mathcal{D} \to \mathbb{R}$ be a function. The upper contour set of $\phi$ at $a \in \mathbb{R}$ is the set

$$U_\phi(a) = \{ x \in \mathcal{D} : \phi(x) \geq a \}.$$  

\footnote{If this prospect does not contain the zero-payoff, we add it with probability zero.}
A SOLUTION CONCEPT FOR GAMES WITH ALTRUISM AND COOPERATION

$g$ is called quasiconcave on $\mathcal{D}$ if $U_\phi(a)$ is a convex set for all $a \in \mathbb{R}$.

The following definition appeared in [Cr90], Definition 3. In this definition the word game is used to denote a classical finite game in normal form $\mathcal{G} = \mathcal{G}(P, S, u)$, where the utility functions are extended to the mixed strategies in a possibly non-linear manner.

**Definition 9.3.** The convexified version of a game is obtained from the game by replacing each player’s preferences by the quasiconcave preferences whose upper contour sets are the convex hulls of his original upper contour sets, leaving other aspects of the game unchanged.

We now define Crawford’s equilibria in beliefs through an equivalent condition proved by Crawford himself in [Cr90], Theorem 1.

**Definition 9.4.** An equilibrium in beliefs is any Nash equilibrium of the convexified version of the game.

Crawford proved in [Cr90], Observation 1, that a Nash equilibrium is always an equilibrium in beliefs and, in Observation 2, that the set of equilibria in beliefs coincides with the set of Nash equilibria if the players have quasiconcave preferences.

We can now define the cooperative equilibria of a game in explicit form.

**Definition 9.5.** The cooperative equilibria of a game in explicit form $\mathcal{G} = \mathcal{G}(P, S, g, g, a, f)$ are obtained applying to the reduced game $\text{Red}(\mathcal{G})$ the procedure described in Section 4, replacing

- the function $g_i(\sigma)$ with the function $V_i(\sigma)$,
- the notion of Nash equilibrium with the notion of equilibrium in beliefs,
- the value function $v_i(p)$ in (1) with the one in (15).

**Theorem 9.6.** Cooperative equilibria exist for all finite games in explicit form.

**Proof.** Let $\mathcal{G} = \mathcal{G}(P, S, g, g, a, f)$ be a finite game in explicit form. We have already proved in Section 8 that the iterated deletion of strategies leads to a well defined and non-empty subgame $\text{Red}(\mathcal{G})$. We shall prove that the construction in Section 4 can be applied to $\text{Red}(\mathcal{G})$.

Fix a coalition structure $p$ and let $\mathcal{G}_p$ be the game obtained by $\text{Red}(\mathcal{G})$ grouping together the players in the same coalition, as in Equation (2). By Crawford’s theorem (see [Cr90], Theorem 2), the set of equilibria in beliefs of $\mathcal{G}_p$ is not empty. Indeed, this is just the set of Nash equilibria of the convexified game. Now, since the preferences in cumulative prospect theory are described by a continuous function and since continuity is preserved by passing to the convexified version (see [Ro70], Theorem 17.2), it follows that the set of equilibria in beliefs of $\mathcal{G}_p$ is compact. Consequently, the sets $M(p, \alpha, p)$ in Equation (3) are non-empty and the definition of the induced game $\text{Ind}(\mathcal{G}, p)$ goes through. Observe that the induced game is not empty, since the value of a prospect is at most as the maximal outcome of the prospect, which is an infimum of values attained by the function $v \circ V_i$. Consequently, the set of $\sigma$‘s such that $(v \circ V_i)(\sigma) \geq v_i^{\text{CPT}}(\sigma)$ is non-empty. Consequently, the set of mixed strategies of the induced game is a non-empty convex and compact subset of the set of mixed strategies of the original game $\mathcal{G}$. Since in the convexified version of a game the set of mixed strategies does not change, the convexified version of $\text{Ind}(\mathcal{G}, p)$ has a non-empty set of Nash equilibria (Indeed, observe that Nash’s proof of existence of equilibria goes through also if only distinguished convex and compact subsets of mixed strategies are allowed). Applying Theorem 1 in [Cr90],
it follows that the induced game Ind(\(G, p\)) has a non-empty set of equilibria in beliefs. Hence, Definition 4.12 defines a non-empty notion of equilibrium.

Consequently, Definition 9.5 defines a non-empty notion of equilibrium. □

The following corollary follows straight from the construction.

**Corollary 9.7.** The exact cooperative equilibrium of a game \(G\) does not depend on the fairness functions and on the altruism parameters, if

1. \(G\) does not have any super-dominated strategies,
2. for every coalition structure \(p\), the game \(G_p\) has a unique equilibrium in beliefs.

**Remark 9.8.** Also in this case we may define the quantal cooperative equilibrium under cumulative prospect theory and taking into account altruism: agent \(i\) plays with probability \(e^{v_{CPT_i}(p)} / \sum_p e^{v_{CPT_i}(p)}\) a quantal level-k solution of the induced game Ind(Red(\(G\)), \(p\)). Such quantal cooperative equilibrium explains deviations from Nash equilibrium that have been observed also in purely competitive games, as the asymmetric matching pennies experimented in [Go-Ho01], that is, the game with gains:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>320,40</td>
<td>40,80</td>
</tr>
<tr>
<td>D</td>
<td>40,80</td>
<td>80,40</td>
</tr>
</tbody>
</table>

It was reported in [Go-Ho01] that most of vertical players played the strategy \(U\) and most of the horizontal players played the strategy \(R\). Observe that the Nash equilibrium for the vertical player is the uniform measure on \(\{U,D\}\), since the gains of the horizontal player are the same as in the matching pennies. We believe that this behavior ultimately relies in a mistake of the vertical players due to the illusion of a large gain and this mistake is predicted by the horizontal player. This interpretation is confirmed by the cooperative equilibrium. Indeed, the value of the cooperative coalition is easily seen to be equal to 40 for both players and therefore, exact cooperative equilibrium reduces to the Nash equilibrium and quantal cooperative equilibrium reduces to the quantal level-k solution. The latter one performs well in such a situation: if the vertical player makes the mistake to think that the horizontal player is level-0 and then she or he is indifferent between playing \(L\) and \(R\), then the vertical player would have a strong incentive to play the strategy \(U\). At this point, the assumption that the horizontal player is level-2 implies that she or he best responds (up to a small mistake) to the strong deviation towards \(U\), which is a strong deviation towards \(R\).

10. **Summary, conclusions and open problems**

Over the last decades it has been realised that all classical solution concepts for one-shot normal form games fail to predict human behavior in several strategic situations.

The purpose of this paper was to reconduct these failures to two basic problems, the use of utility functions and the use of solution concepts that do not take into account human attitude to cooperation. While the former problem could be theoretically overcome replacing utility functions by gain functions and applying cumulative prospect theory, the second problem needs a different analysis of the structure of a game. We founded this new analysis on a seemingly reasonable principle of cooperation.

(C) Players try to forecast how the game would be played if they formed coalitions and then they play according to their best forecast.
To make this idea formal, it has required some time. In Section 2 we have observed
that passing from utility functions to gain functions implies that we must take into
account new phenomena, such as altruism and perception of gains. We have formal-
ized these phenomena defining the so-called games in explicit form. After an example
describing informally the main idea, in Section 4 we have formalized the principle
of cooperation and we have defined the cooperative equilibrium for games in explicit form
without using altruism parameters and cumulative prospect theory. The reason of this
choice is that altruism and cumulative prospect theory play an active role only on a
limited class of games. Indeed, in Section 5 we have shown that the cooperative equi-
librium without altruism and cumulative prospect theory already performs well in a
number of relevant games. In Section 6 we have discussed a few examples where cumu-
lative prospect theory starts playing an active role and, after a short introduction to
cumulative prospect theory in Section 7 we have started to adapt the definition given in
Definition 4.12 in order to be applied to every game in explicit form and using cumula-
tive prospect theory. In Section 8 we have used altruism parameters to delete strategies
that are not good for the collectivity. This iterated deletion of strategies leads to de-
fine a certain subgame. The study of this subgame (done in Section 4 under expected
utility theory and in Section 9 under cumulative prospect theory) contains all relevant
new ideas of the paper, that are, the use of the principle of cooperation and the use
of cumulative prospect theory: we have assumed that every players try to forecast how
the game would be played if they formed coalition; we have used ideas from cumulative
prospect theory to define a notion of value of a coalition and then, appealing to some
Bernoulli-type principle, we have postulated that agents play according to the coalition
with highest value.

As shown in the examples in Section 5, the theory has many positive consequences:
to the best of our knowledge, it is the first theory able to predict human behavior in the
parametrised Traveler’s Dilemma, parametrised Prisoner’s Dilemma, Nash bargaining
problem, Bertrand competition, ultimatum game, public good game. It also performs
well in Hawk-Dove and in other specific games where previous theories fail to predict
human behavior. These successful applications and the lack of examples where the coop-
erative equilibrium fails (qualitatively) to predict human behavior, make us optimistic
about this direction of research. Nevertheless, we are perfectly aware that the theory is
questionable in several points which deserve more attention in future researches. These
points include:

1. The formula used in Equation 1 to compute the value of a coalition seems
a quite reasonable one and it meets the experimental data quite well, but it
is certainly only a first tentative. More thoughts, possibly supported by more
experimental data, may help to understand the value of a coalition. The main
point is probably:
   • to understand whether the value should be computed taking into account
     also deviations towards safe strategies, as sketched in Example 6.2 and not
     only taking into account deviations to achieve a larger gain, as we did in
     this paper.

2. The exact computation of the cooperative equilibrium is hard for several rea-
sons. First because it goes through the computation of the equilibria in be-
liefs of several (sub)games. These equilibria are computationally hard to find

\[\text{As observed by J. Halpern in a private communication, it is implausible that an agent would}
\text{consider all coalitions. In even moderately large games, there are just too many of them. She may}\]
Second, because it uses cumulative prospect theory, that is computationally harder than expected utility theory. On one hand, the method that we have proposed is perfectly algorithmic and therefore it might be helpful to write a computer program to compute the cooperative equilibria and make easier the phase of test them on easy real-life situations. On the other hand, it would be important to investigate some computationally easier variant. Of course, quantal level-k theory can be seen as a computationally easier variant, but this theory has the negative-counter-part that it is not predictive, in the sense that one has to make an experiment to estimate the error parameter. One could try to avoid this problem using the level-k theory (i.e., only bounded rationality).

(3) Iterated deletion of strategies using altruism parameters in Section 8 was certainly quite sketchy and it is possible the future researches will suggest a different procedure.

Open problems include:

(1) Many experiments with different purposes should be conducted. An interesting fact is that cooperative equilibrium makes sometimes predictions that are completely different from the standard theories. A stream of experiments should be devoted to verify or falsify these predictions. For instance,

- Apparently, the cooperative equilibrium is the unique solution concept predicting a tendency to cooperation in the public good game, as the marginal return approaches 1. It seems that this prediction has a partial confirmation from experimental data, but these experimental data have been collected mainly using relatively small marginal returns and therefore, more precise experiments should be conducted.
- Apparently, the cooperative equilibrium is the unique solution concept predicting a rate of cooperation in the Prisoner’s dilemma depending on the particular gains. Experiments with parametrized Prisoner’s dilemma should be conducted to verify or falsify this prediction.

Another stream of experiments should be devoted to answer some theoretical questions. At this first stage of research, we believe that the most important one is:

- to understand whether the value should be computed taking into account also deviations towards safe strategies, as sketched in Example 6.2, and not only taking into account deviations to achieve a larger gain, as we did in this paper. This could be clarified by conducting experiments on parametrized generalized coordination games as in Example 6.2.

(2) Have a better understanding of the relation between Nash equilibria and cooperative equilibria (under expected utility theory) for two-person zero-sum games, when the players have the same perception of gains. Indeed, Nash equilibrium performs quite well for zero-sum games and it is possible that all deviations from Nash equilibrium can be explained only making use of cumulative prospect theory. Therefore, it would be important to understand if the cooperative equilibrium (under expected utility theory and assuming $f_1 = f_2$) refines Nash equilibrium, in the sense that the set of exact cooperative equilibria is always a subset of the set of Nash equilibria. Indeed, we have seen in Examples 6.3 and 8.12 and also in some non-zero-sum examples, such as Example 6.2, that consider some natural coalitions (e.g., the coalition of all agents), but only a relatively small number.

Of course, a theory characterizing which coalitions would be considered is not easy to come by.
sometimes the cooperative equilibrium selects the \textit{most plausible} Nash equilibrium. Is this always true? What is \textit{most plausible} formally stand for? In this context, it would be interesting to start from relevant classes of zero-sum games, as the group games, introduced and studied in [Mo10], [Ca-Mo12], [Ca-Sc12]. Of course, also a counter-example would be very important to understand if and where the theory can be modified.

(3) From a theoretical point of view, it would be interesting to find an axiomatic definition of the value of a coalition; namely, a set of reasonable and simple axioms à la Shapley implying that the value of a coalition (under expected utility theory) must have the shape

\[
\sum_{J \subseteq P \setminus \{i\}} e_{i,J}(p) \tau_{i,J}(p).
\]

(4) A seemingly interesting open problem is the development of a theory for iterated versions of one-shot normal form games. It is likely that, in this case, the function \( \tau_J \) depends also on the period \( n \). The idea indeed is that a player may trust less or more about a coalition and a particular set of players \( J \) depending on how those players behaved in the preceding periods. Another reason of interest about this problem is that this seems to be the right place to unify the cooperative equilibrium with the solution concept introduced in [Ha-Pa11], using algorithmic rationality. The idea behind this latter solution concept, built on previous works by Neyman [Ne85] and by Rubinstein [Ru86], is that computation is costly and therefore players prefer to play \textit{easy strategies} and, in iterated versions of a one-shot game, they prefer not to change strategy, in order to avoid additional (costly) computations. From a technical point of view, the equilibrium with costly computation is defined exactly as the Nash equilibrium, but using a different utility function, taking into account the cost of a strategy. This notion of equilibrium can indeed explain quite well deviations towards cooperation in the iterated Prisoner’s dilemma and in the iterated Traveler’s dilemma (for instance, the tit-for-tat strategy in the Prisoner’s dilemma is intuitively very little costly, since players have to remember only what happened in the previous iteration). Nevertheless, this solution concept performs very bad in one-shot games. To see this, observe that all pure strategies must have the same cost and this cost must be minimal. Therefore, applying this solution concept to Nash equilibrium, one finds that one-shot games with a pure Nash equilibrium have the property that the equilibrium with cost computation coincides with the Nash equilibrium. Consequently, this solution concept contradicts the experimental data collected in one-shot versions of the Prisoner’s and the Traveler’s dilemma. This suggests that Halpern-Pass’ solution concept using algorithmic rationality must be applied to some other solution concept. In a future stage of this research, it would be then interesting to try to apply it to the cooperative equilibrium.

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