Reading: Strogatz, sections 6.0-6.5; Liz's notes on the variational equation; Parker&Chua, Appendix B.

Bibliography:

- D. Ruelle, “Strange Attractors,” *The Mathematical Intelligencer*, **2**:126-137, 1980. There’s a closely related paper by him in the Cvitanovic collection, which is on library reserve.

Problems:

1. \( \text{[math]} \) Derive the Jacobian \( D_{\vec{x}}\vec{F} \) for the Lorenz system:

\[
\vec{F}(\vec{x}, a, r, b) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix}
\]  

2. \( \text{[math]} \) Write down the associated variational system \( \dot{\delta} = D_{\vec{x}}\vec{F}\delta \). The product of the Jacobian matrix \( D_{\vec{x}}\vec{F} \) and the \( n \times n \) matrix of variations

\[
\dot{\delta} = \begin{bmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{bmatrix}
\]

yields an \( n \times n \) matrix of the derivatives of the variations \( \dot{\delta} \). \( \delta_{xy} \), for instance, is the component of the \( x \)-variation that comes from the previous \( y \)-variation, and \( \dot{\delta}_{xy} \) is its rate of change with time.
3. *programming* A combination of this variational derivative and the original system derivative from PS5 can be used to integrate the \((n^2+n)\)-dimensional variational equation

\[
\begin{bmatrix}
\dot{x} \\
\dot{\delta}
\end{bmatrix} = \begin{bmatrix}
F \\
D_x F \cdot \delta
\end{bmatrix}
\]

from the initial condition

\[
\begin{bmatrix}
\vec{x}_0 \\
I
\end{bmatrix}
\]

with \(t = t_0\). The time evolution of the first \(n\) elements of this set of initial conditions follows the trajectory \(\phi_t(\vec{x}_0, t_0)\). The row and column sums of the matrix formed by the next \(n^2\) elements are different ways to look at the evolved versions of the initial variations: the first column sum, for example, gives the \(x\) component of the evolved variation, while the first row sum tells you what the \(x\)-piece of the *original* variation has grown into.

Integrate the Lorenz variational equation using RK4 (not ARK4) from the following initial conditions for 100 steps. Use \(a = 16\), \(r = 45\), \(b = 4\), and a timestep of .001. In each case, give

- the components of the evolved matrix \(\delta\)
- the evolved variations (the column sums of \(\delta\))

*at the endpoint of the trajectory.* Use \(t_0 = 0\) (this is really immaterial, as the system is autonomous). You need only turn in these twelve numbers for each of the following questions; no plots or intermediate values are necessary.

(a) \([x \ y \ z \ \delta_{xx} \ \delta_{xy} \ \delta_{xz} \ \delta_{yy} \ \delta_{yz} \ \delta_{zx} \ \delta_{zy} \ \delta_{zz}] = [0 \ 1 \ 2 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1]\)

(b) \([10 \ -5 \ 2 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1]\)

(c) \([0 \ -1 \ 2 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1]\)

(d) Look carefully at the evolved matrices of variations and describe some of their interesting features. From which point [(a), (b), or (c)] do the variations grow fastest? In which direction \((x, y, z)\)? Do you notice any symmetries or gross differences between the different points (e.g., “the \(y\)-variation grows really fast near point A, but less so near point B...”)?