University of Colorado
Department of Computer Science

Chaotic Dynamics – CSCI 4446/5446
Spring 2012

Problem Set 4

Issued:	7 February 2012
Due:	14 February 2012

Reading: *Strogatz, sections 2.0-2.3, 2.8, 6.0-6.5, 6.7 and chapter 5; sections 1 and 2 of Liz's ODE Notes;* Parker&Chua, chapter 4; Lorenz's "Computational Chaos..." article (listed below). Section 2 of Liz's TSA Notes may also be useful; see the course webpage.

Bibliography:

- D. Auerbach, P. Cvitanovic, J.-P. Eckmann, G. Gunaratne and I. Procaccia, "Exploring chaotic motion through periodic orbits," *Phys. Rev. Lett.* **58**:2387-2389 (1987). There's another UPO paper by Cvitanovic in the Campbell reprint collection, but it's heavy going.
- E. Lorenz, "Computational Chaos – A Prelude to Computational Instability," *Physica D*, **35**:299-317 (1989).
- J. Marsden *et al.*, "Symmetry, Stability, Geometric Phases, and Mechanical Integration," *Nonlinear Science Today*, volume 1, 1991.
- R. H. Miller, "A Horror Story about Integration Methods," *J. Computational Physics* **93**:469-476 (1991).
- W. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing*, Cambridge, 1988.
- D. Tritton, "Chaos in the Swing of a Pendulum," *New Scientist*, 7/24/86.
- H. C. Yee *et al.*, "Dynamical approach study of spurious steady-state numerical solutions of nonlinear differential equations. 1. The dynamics of time discretization and its implications for algorithm development in computational fluid dynamics," *Journal of Computational Physics* **97**:249-310 (1991)

Problems:

1. Write a fourth-order fixed-time-step Runge-Kutta integrator *from scratch.* Inputs should consist of a starting time t_0, time step Δt, number of steps n, and starting value $\vec{x}(t_0)$ for the state vector. If your language allows functions as arguments, make the system derivative an argument as well. The output of this procedure — a series of state vectors representing the n-point state-space trajectory emanating from $\vec{x}(t_0)$ — should go to a file.

Please do not use any canned numerical integration routines, commands, functions, etc. Later problem sets will use this integrator with many different systems; it will save you a **lot** of time later if you write this version to take arbitrary-size state vectors!

The rest of the problems in this set concern the following equation:

$$ml\frac{d^2}{dt^2}\theta(t) + \beta l\frac{d}{dt}\theta(t) + mg\sin\theta(t) = A\cos\alpha t$$

This is the equation of motion of a forced, damped pendulum. The state vector is $[\theta, \omega]^T$; the former is measured in radians and the latter in radians per second. Generate your solutions of this equation using your RK4 solver.

2. Use m=0.1kg, l=0.1m, β=0 and set the drive amplitude and frequency to zero ($\alpha = A = 0$).

(a) Turn in a plot of the state-space trajectory emanating from the point $[\theta, \omega] = [3, 0.1]$ with $\Delta t = 0.005$. Is this initial condition near an equilibrium point? Which one? Is that point stable or unstable?

(b) Turn in a plot of the state-space trajectory emanating from the point $[\theta, \omega] = [0.01, 0]$. You'll have to use different coordinate axes from those in part (a) to get a good plot. Does this trajectory look more like a perfect ellipse than the trajectory of part (a)? *CSCI 5446 students: why or why not?*

3. Use your integrator to generate a *state-space portrait* of the system, using the coefficient values given in problem 2 above. If you have trouble selecting a representative set of starting points — that is, a set that samples all of the features of the dynamics — look at figure 6.7.3 in Strogatz for an example. Turn in a copy of your plot.

4. Now repeat problem 3 with $\beta = 0.25$. What happens to the various features of the plot? What does this imply about the physical dynamics? Turn in a copy of this plot. What do you think would happen with a higher β? What about a lower β?

5. Modify your code so that it plots θ modulo 2π and see what that does to your results in problem 4. (Here are some examples that should help you understand what "modulo" means: 2 modulo 3 = 2; 3 modulo 3 = 0; 4 modulo 3 = 1; 7 modulo 3 = 1.) Make sure you understand this, and turn in a copy of the plot.

6. Leaving β at 0.25, turn on the drive. Vary the drive frequency α and amplitude A (i.e., "explore the parameter space") and describe and explain what you see on the plots — in the vocabulary of nonlinear dynamics, not physics (e.g., "bifurcation," etc.) Find a chaotic orbit[1] and turn in a plot of it. Hint: start with the drive frequency at about 3/4 of the natural frequency of the device and slowly increase the drive amplitude. (The natural frequency is related to g and l in the manner derived during the first week of the semester.) Make sure you plot θ modulo 2π!

7. Turn the drive back off, set β back to 0, play with the timestep, and describe the effects (Hint: try increasing the timestep until weird things happen. Describe and attempt to explain them.)

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[1] "orbit" and "trajectory" are synonyms