How to Make a Square Grid Framework with Cables Rigid

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Abstract

This paper solves the problem of making a bipartite digraph strongly connected by adding the smallest number of new edges that preserve bipartiteness. A result of Baglivo and Graver shows this corresponds to making a 2-dimensional square grid framework with cables rigid by adding the smallest number of new cables. We prove a min-max formula for the smallest number of new edges in the digraph problem, and give a corresponding linear-time algorithm. We generalize these results to the problem of making an arbitrary digraph strongly connected by adding the smallest number of new edges, each of which joins vertices in distinct blocks of a given partition of the vertex set.

1 Introduction

In the *connectivity augmentation problem* we are given a graph and a positive integer k. We wish to find a smallest set of edges whose addition makes the graph k-connected. Here "k-connected" can refer to vertex- or edge-connectivity. Practical applications include the design of reliable networks ([8], [15]). Much of the work on connectivity augmentation is discussed in the survey article [7]. The result most relevant to this paper is one of the first augmentation algorithms, due to Eswaran and Tarjan, which makes a digraph strongly connected in linear time [6].

Restricted versions of the augmentation problem may be even more interesting in terms of both theory and practical applications. Many restrictions can be modeled by the minimum cost augmentation problem. However even special cases of this problem are NP-hard, e.g., making a digraph strongly connected by adding the smallest number of edges from a given set [6]. Other restrictions that have been investigated include the following: Augmenting to achieve k-edge-connectivity while preserving simplicity of the given graph [2],[11]; augmenting while preserving planarity [12], [13]; augmenting an undirected bipartite graph to achieve 2-vertex-connectivity [9] or k-edge-connectivity [3] while preserving bipartiteness. Augmenting a biconnected graph to achieve both k-edge connectivity and 3-vertex connectivity is studied in [10].

This paper investigates the problem of making a bipartite digraph strongly connected while preserving bipartiteness. We prove a min-max formula for the smallest possible number of new edges. We give a corresponding linear-time algorithm. In addition we generalize these results to the problem of strong-connectivity augmentation with partition constraints (defined precisely below). We now give some motivation for the problem.

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A basic 2-dimensional structure in statics is the *square-grid framework* (see Fig. 1). It consists of horizontal and vertical *rods*. Each rod has the same length, and is rigid. Incident rods are connected by a *joint* which allows the rods to pivot. The rods collectively join all grid-points that are in a rectangular region and are adjacent horizontally or vertically. In addition there may be a number of "diagonal rods" that join two diagonally-opposite points of a grid square. Such a framework is *rigid* if it has no nontrivial deformations, i.e., fixing the position of one rod in the plane, the positions of all other rods are uniquely determined. (These concepts are described more precisely in [14].) Bolker and Crapo [4] show that any square-grid framework has a natural representation as a bipartite graph, and the framework is rigid if and only if its bipartite graph is connected (Fig. 1). Because of this relation the results of [3] answer questions such as, find a smallest set of new rods whose addition results in a framework that is rigid, even after an arbitrary set of k diagonal rods fail.

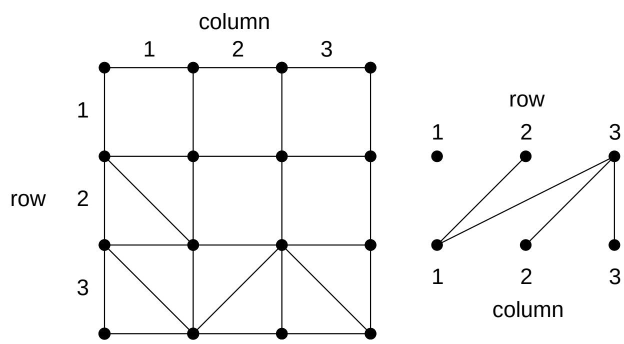

Figure 1: Row 1 can pivot in this square grid framework. But if we add a rod in row 1, column 3 the framework becomes rigid and the bipartite graph becomes connected.

A potentially cheaper way to make a framework rigid is to use pliable material along the diagonals. A *cable* can be compressed but not stretched, and can join two diagonally-opposite points of a grid square (see Fig. 2). Baglivo and Graver [1] show that a framework with cables has a natural representation as a bipartite digraph, such that the framework is rigid if and only if its bipartite digraph is strongly connected (Fig. 2). The *cable-framework rigidity problem* is to make a given framework with diagonal cables rigid by adding the smallest number of new cables. Our solution to the graph augmentation problem solves the cable-framework rigidity problem, by the theorem of Baglivo and Graver.

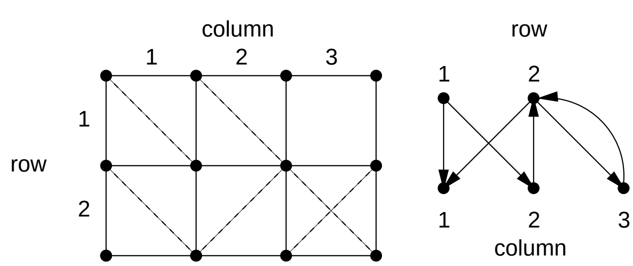

Figure 2: Here dotted lines represent cables. The joint at the upper left corner can move inwards, compressing the cables in row 1, columns 1 and 2, and row 2, column 1. But if we rotate the cable in row 1, column 1 by 90° the framework becomes rigid and the bipartite digraph becomes strongly connected.

The main part of our solution to the problem of bipartite strong-connectivity augmentation is a min-max formula for the optimum number of new edges (Lemma 5.3). This formula extends to the general problem of strong-connectivity augmentation with partition constraints (Theorem 7.8). Roughly put, the formula states that the optimum number of new edges either equals a natural lower bound or is one more than that. The latter holds exactly when the graph belongs to one of four infinite families.

This result has similar overall structure to the min-max formula of [3] for bipartite undirected graph k-edge-connectivity augmentation: Both formulas have an infinite number of exceptional graphs that require one extra edge. The formula of [3] has a single family of exceptional graphs. Our problem, just for the $k = 1$ case, seems to have more complicated structure, with four distinct families. Together these two results give evidence that min-max formulas with exceptions may be a general phenomenon.

Our result corresponds to the $k = 1$ case of [3]. This case is the problem of augmenting a bipartite graph to make it connected and bipartite. This simple problem has no exceptional families. It is equivalent to the rod-framework rigidity problem, which was solved in [1] and [14]. Until now the cable-framework rigidity problem has been left open, perhaps because the phenomenon of exceptional families has been ignored.

Section 2 of this paper gives basic facts about our augmentation problem, including lower and upper bounds on the solution size. Sections 3–5 prove the min-max formula for bipartite augmentation. Sections 6 – 7 prove the formula in general (reducing to the case covered in Section 3). Section 8 presents efficient algorithms for the augmentation problems. Section 9 discusses some augmentation problems related to ours.

The material in Section 5 is not logically necessary for proving the general formula. However it gives a simpler proof of the special case needed for the cable-framework rigidity problem. Furthermore this proof is the basis of the algorithm in Section 8 that is more efficient than the general algorithm. The reader should also be aware at the outset that many details of the algorithms in Section 8 are omitted because they directly follow the proofs in preceding sections.

This section closes with some terminology. In the problem of *(strong-connectivity) augmentation with partition constraints* we are given a digraph $D = (V, E)$ together with a partition $\mathcal{P}$ of V into disjoint sets called *color classes*. We wish to make D strongly connected, if possible, by adding the smallest number of new directed edges, where no new edge joins vertices in the same color class. The *bipartite strong-connectivity augmentation problem* is the special case where D is a bipartite digraph and the color classes are the two sets of the given bipartition of V.

We use B, W ("black, white") as typical colors. A vertex v is *colored* B if $v \in B \in \mathcal{P}$. A *singleton vertex* v has $\{v\} \in \mathcal{P}$, i.e., no other vertex has its color. There is no restriction on a new edge incident with a singleton vertex. Note that in our general augmentation problem there is no restriction on an original edge of E: it can join vertices of the same color. We will refer to the augmentation problem for a graph D without mentioning $\mathcal{P}$ when $\mathcal{P}$ is clear from context.

The symbols $\subseteq$ and $\subset$ denote set containment and proper set containment respectively. An overbar denotes set complement (with the universe understood), e.g., $\overline{B}$ denotes the set of non-black vertices $V - B$.

Consider a digraph $G = (V, E)$. A set consisting of a single vertex x is simply denoted by x. If S is a set of vertices containing y but not x then an edge (x, y) *enters* S and an edge (y, x) *leaves* S. A vertex x *precedes* a vertex y (alternatively, x *can reach* y) if there is a path from x to y.

A digraph with no cycles is a *dag* (directed acyclic graph). A *source* (*sink*) is a vertex with no entering (leaving) edge. An *extreme* vertex is a source or a sink. The *strong component graph* of a digraph is formed by contracting every strongly connected component. It is a dag.

2 Basic facts

This section presents basic concepts and notation. In particular it defines the quantity Φ and proves that an optimum solution to our augmentation problem uses either Φ or $\Phi+1$ new edges.

The special case of strong-connectivity augmentation with partition constraints when D is a dag is called the *dag problem*. The general problem reduces to the dag problem, as follows. Let D be a given digraph. Let H be its strong component graph. To assign colors to the vertices of H let a vertex x of H correspond to the strong component X of D. If every vertex of X is colored B then x is colored B. If X contains vertices of two or more colors then x is a singleton vertex. Solutions to the augmentation problem with partition constraints on D and H correspond because of the following observation: For two vertices x_i, $i=1,2$ of H corresponding to strong components X_i, $i=1,2$ of D, x_1 and x_2 have different colors if and only if some vertex of X_1 and some vertex of X_2 have different colors.

From now on we will work on the dag problem for a dag $H=(V,E)$. We use the following notation. OPT is the smallest number of new edges needed to solve the given dag problem. By definition $OPT=0$ if $|V|=1$ and $OPT=\infty$ (i.e., there is no solution) if $|V|>1$ and all vertices have the same color.

Let X be a set of vertices of H. X^+ denotes the set of all sources in X and X^- the set of all sinks. For instance V^+ is the set of all sources of H. An isolated vertex belongs to $V^+\cap V^-$.

$R^-(X)$ denotes the set of all sinks that can be reached from some source of X. Similarly $R^+(X)$ denotes the set of all sources that can reach some sink of X. For $X,Y\subseteq V$, the predicate $r(X,Y)$ is true when some source of X precedes some sink of Y. This is equivalent to $R^-(X)\cap Y\neq\emptyset$. Note as a special case if x is a source and y is a sink, $r(x,y)$ means x precedes y. We abbreviate $r(X,X)$ to $r(X)$. Negation is denoted by $\neg$, e.g., $\neg r(X,Y)$.

We will characterize OPT in terms of the following quantity:

$$\Phi=\begin{cases} 0 & \text{if } |V|=1\\ \infty & \text{if } |V|>1 \text{ and } |\mathcal{P}|=1\\ \max\{|V^+|,\ |V^-|,\ |B^+|+|B^-| : B\in\mathcal{P}\} & \text{otherwise.}\end{cases}$$

Clearly $OPT=\Phi$ in the first two cases. In the third case $OPT\geq\Phi$: A solution must contain an edge entering every source and an edge leaving every sink. This implies $OPT\geq|V^+|,|V^-|$. One new edge cannot leave a sink and enter a source of the same color. This implies $OPT\geq|B^+|+|B^-|$ for any color B.

There are simple graphs with $OPT>\Phi$. For instance the graph on four vertices $b^+\in B^+$, $b^-\in B^-$, $w^+\in W^+$, $w^-\in W^-$ with edges (b^+,w^-) and (w^+,b^-) has $\Phi=2$ and $OPT=3$.

Any instance of the dag problem has a *directionally symmetric* instance, obtained by reversing the orientations of all edges. Solutions to these two problems correspond, and OPT and Φ are the same. We invoke directional symmetry to simplify the proofs.

It is convenient to work with graphs that have no isolated vertices. If $|V|>1$ we eliminate an isolated vertex x by replacing it with vertices x_1,x_2 having the same color as x, and an edge (x_1,x_2). (There is no edge (x_2,x_1) so the transformed graph is a dag.) A solution on the transformed graph gives a solution on the original graph by contracting x_1,x_2 to a vertex we call x.

Proposition 2.1 *The transformation to eliminate an isolated vertex does not change the value of* $|V^+|$, $|V^-|$, $|B^+|+|B^-|$ *for* $B\in\mathcal{P}$, Φ *or* OPT.

Proof: We show OPT does not change. Let H be the original graph and H' the transformed graph. Clearly a solution on H' gives a solution on H that is no larger. For the converse let F be a set of new edges in H and F' the corresponding set of new edges in H'. (An edge $(y, x) \in F$ corresponds to edge (y, x_1) and $(x, y) \in F$ corresponds to (x_2, y).) Assume $H + F$ is strongly connected. Let y be a vertex of $H - x$ (y exists by the assumption $|V| > 1$). In $H + F$, y can reach every other vertex and every other vertex can reach y. Clearly the same holds in $H' + F'$. $\square$

From now on assume the given dag H has no isolated vertices. We drop this assumption in the statement of our main results (Lemma 5.3 and Theorem 7.8).

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In a *unicolored dag* the extreme vertices all have the same color. In a *bicolored (tricolored) dag* the extreme vertices are contained in at most two (three) color classes, which we take as B and W (and G).

Lemma 2.2 *A unicolored dag has $OPT = \Phi$.*

Proof: Assume $|\mathcal{P}| > 1$ else obviously $OPT = \Phi = \infty$. Thus $\Phi = |V^+| + |V^-|$. Choose any vertex y of a color different from the extreme vertices. Add edges (y, x^+) for every $x^+ \in V^+$ and (x^-, y) for every $x^- \in V^-$. Thus y can reach every source and every sink can reach y. This implies y can reach every vertex and every vertex can reach y. $\square$

A *rooted dag* is a dag with exactly one source or exactly one sink.

Lemma 2.3 *Any rooted dag has $OPT = \Phi$.*

Proof: Assume the dag H is not unicolored. By symmetry assume H has exactly one source x^+. Thus $\Phi = |V^-|$.

Suppose x^+ is a singleton vertex. Adding an edge from each sink to x^+ makes H strongly connected. Hence $OPT = \Phi$.

Suppose x^+ is black. Some sink y^- is not black. Add edge (y^-, x^+). In the new graph H', x^+ and y^- are in the same strong component. If H' is strongly connected we are done. Otherwise the strong component graph of H' is rooted, with a unique source that is a singleton vertex. Also Φ has decreased by 1. So the lemma follows from the previous case. $\square$

A set of extreme vertices is *consistently-colored* unless it contains a source and sink of the same color. The dag is *consistently-colored* if $V^+ \cup V^-$ is consistently-colored (equivalently any color B has B^+ or B^- empty).

Definition 2.4 *A pair of vertices $x^- \in V^-$ and $y^+ \in V^+$ is* compatible *if the pair is consistently-colored and there are vertices $x^+ \in V^+ - y^+, y^- \in V^- - x^-$ with $r(x^+, x^-)$ and $r(y^+, y^-)$.*

Lemma 2.5 *Any consistently-colored set of two sources and two sinks contains a compatible pair.*

Proof: Let the consistently-colored set consist of sources y_i^+, $i = 1, 2$ and sinks x_i^-, $i = 1, 2$. For $i = 1, 2$, if x_i^-, y_i^+ is not compatible then $r(y_i^+, x_i^-)$. This makes x_1^-, y_2^+ compatible. $\square$

If vertices $x^- \in V^-$, $y^+ \in V^+$ form a compatible pair, *augmenting* x^-, y^+ means adding a new edge (x^-, y^+) to the graph and passing to the new strong component graph. Thus if $H + (x^-, y^+)$ has a strong component containing x^- and y^+, we contract it. The augment operation also adds the new edge (x^-, y^+) to the solution set of the augmentation problem.

Lemma 2.6 *Augmenting a compatible pair x^-, y^+ decreases $|V^+|$, $|V^-|$ and $|B^+|+|B^-|$ for B the color of x^- or y^+, each by one. Furthermore it does not create an isolated vertex.*

Proof: Let H be the given graph and let $H' = H + (x^-, y^+)$. We consider two cases.

Suppose $\neg r(y^+, x^-)$ in H. Then H' is a dag. y^+ is not a source of H' since it has an entering edge. Similarly x^- is not a sink of H'. The lemma follows.

Suppose $r(y^+, x^-)$ in H. Hence our two vertices are in the same strong component of H'. Let this strong component correspond to vertex X of the strong component dag K of H'. X is not a source of K, since another source precedes it (specifically, the source x^+ from Definition 2.4). Similarly X is not a sink of K. Now the lemma follows as above. $\square$

Lemma 2.7 *Any consistently-colored dag has $OPT = \Phi$.*

Proof: A consistently-colored dag has $\Phi = \max\{|V^+|, |V^-|\}$. Without loss of generality let $\Phi = |V^-|$. If the given graph H is rooted we are done by Lemma 2.3. If not, H has a compatible pair by Lemma 2.5. Augment it. Lemma 2.6 shows the graph remains consistently-colored and Φ decreases by 1. Repeating this procedure eventually makes H rooted. $\square$

Lemma 2.8 *Let H be a dag with no compatible pair of vertices. Then H is either rooted, unicolored or bicolored. In the third alternative if H is bicolored by B and W then H contains two subgraphs that are rooted dags, one spanning $B^+ \cup W^-$, the other spanning $W^+ \cup B^-$, and $B^+, B^-, W^+, W^- \neq \emptyset$.*

Proof: Suppose H is neither rooted nor unicolored. We show the third alternative of the lemma holds.

If H is consistently-colored then Lemma 2.5 with H not rooted gives a compatible pair. We conclude that some color B has $B^+, B^- \neq \emptyset$.

Claim: $\overline{B}^- \neq \emptyset$ implies H contains a rooted dag spanning $B^+ \cup \overline{B}^-$, and furthermore $\overline{B}^+ \neq \emptyset$.

Proof: The hypothesis and Lemma 2.5 imply at least one of the sets B^+, $\overline{B}^-$ has exactly one vertex. First suppose $B^+ = \{b^+\}$. Since H is not rooted, $\overline{B}^+ \neq \emptyset$. Since any sink $\overline{b}^- \in \overline{B}^-$ is incompatible with b^+, $r(b^+, \overline{b}^-)$.

Next suppose $\overline{B}^- = \{\overline{b}^-\}$ and $|B^+| > 1$. Since any source b^+ is incompatible with $\overline{b}^-$, $r(b^+, \overline{b}^-)$. Furthermore b^+ does not precede any other sink, which with $B^- \neq \emptyset$ implies $\overline{B}^+ \neq \emptyset$. ♠

Since not all extremes are black, $\overline{B}^+ \cup \overline{B}^- \neq \emptyset$. Now the Claim, along with its directionally-symmetric version, shows H contains two rooted dags, one spanning the extreme vertices of $B^+ \cup \overline{B}^-$ and the other spanning the extremes of $\overline{B}^+ \cup B^-$. It remains only to show that all extreme vertices of $\overline{B}$ have the same color. If they do not, there are vertices of $\overline{B}^+$ and $\overline{B}^-$ with different colors. These vertices are then compatible, a contradiction. $\square$

Lemma 2.9 *Any dag has $OPT \leq \Phi + 1$.*

Proof: First assume the dag H is bicolored. If there is a compatible pair augment it. Lemma 2.6 shows this reduces Φ by one (since H is bicolored). Hence we can assume H has no compatible pairs.

Apply Lemma 2.8. If H is rooted, Lemma 2.3 shows $OPT = \Phi$. If H is unicolored, Lemma 2.2 shows $OPT = \Phi$. Now the third alternative of Lemma 2.8 shows H contains two subgraphs that are rooted dags, one spanning $B^+ \cup W^-$ and the other spanning $W^+ \cup B^-$. We consider two cases that, using directional symmetry, cover all possibilities.

Case 1. $\Phi = |B^+| + |B^-|$.

We have $|B^-| \ge |W^+|$ since $|B^+| + |B^-| = \Phi \ge |V^+| = |B^+| + |W^+|$. Since $W^+ \cup B^-$ is rooted this implies there is a unique white source. Similarly there is a unique white sink.

Add an edge from each black sink to the white sink. The resulting graph has a unique sink, $1 + |B^+|$ sources and $\Phi = 1 + |B^+|$. (Here we use the fact from Lemma 2.8 that $B^+ \neq \emptyset$.) Hence Lemma 2.3 implies the given dag H has $OPT \le \Phi + 1$.

Case 2. $\Phi = |V^-| > |B^+| + |B^-|, |W^+| + |W^-|$.

We have $|B^-| > |W^+|$ since $|B^-| + |W^-| = |V^-| = \Phi > |W^+| + |W^-|$. Thus W^+ consists of a single vertex w^+ which precedes all vertices of B^-. Furthermore B^- contains at least two vertices, say b_1^- and b_2^-. Similarly B^+ consists of a single vertex b^+ which precedes all vertices of W^-, and W^- contains at least two vertices, say w_1^- and w_2^-.

We now show that $OPT = \Phi$ in this case. Add edges (b_1^-, w^+) and (w_1^-, b_1^-). Let K be the strong component dag of the resulting graph. The two new edges decrease $|V^-|$ by two (since w^+ precedes the sink b_2^-). K is a rooted dag, with unique source b^+. Since K has a white sink w_2^-, K has $\Phi = |V^-|$. Now Lemma 2.3 implies $OPT = \Phi$ for both K and H.

Next suppose that H is not bicolored or rooted. First suppose $\Phi > |B^+| + |B^-|$ for every $B \in \mathcal{P}$. Lemma 2.8 shows a compatible pair exists, so choose such a pair and augment it. Lemma 2.6 and our assumption on Φ show the augment decreases Φ by 1. Repeat this as many times as possible. Eventually we get a graph that is bicolored or rooted or has $\Phi = |B^+| + |B^-|$ for some $B \in \mathcal{P}$. It remains only to analyze this last case.

Suppose $\Phi = |B^+| + |B^-|$. Let K be the same dag using just two colors, B and $\overline{B}$. This does not change Φ, i.e., $\Phi \ge |\overline{B}^+| + |\overline{B}^-|$, since $2\Phi \ge |V^+| + |V^-| = |B^+| + |B^-| + |\overline{B}^+| + |\overline{B}^-| = \Phi + |\overline{B}^+| + |\overline{B}^-|$. Since K is bicolored it has $OPT \le \Phi + 1$. Any solution to the augmentation problem on K is a valid solution on H. Hence $OPT \le \Phi + 1$ for H. $\square$

Corollary 2.10 *Any bicolored dag with $\Phi > |B^+| + |B^-|$ for every $B \in \mathcal{P}$ has $OPT = \Phi$.*

Proof: In a bicolored graph an augment decreases all four quantities that define Φ, by Lemma 2.6. Hence a dag satisfying the inequality of the lemma eventually gives a dag that is rooted or satisfies Case 2 (or its directionally-symmetric version) of the above proof. $\square$

This result is generalized by Lemma 3.5.

3 Bicolored dags

This section solves the augmentation problem for bicolored dags. Corollary 2.10 has already shown $OPT = \Phi$ unless Φ is the number of extreme vertices in some color. Hence we will complete the analysis by considering two cases,

$$\Phi = |B^+| + |B^-| > |W^+| + |W^-|;$$
$$\Phi = |B^+| + |B^-| = |W^+| + |W^-|.$$

The main notion for bicolored dags with $OPT > \Phi$ is the 1-blocker. The following definition applies to graphs in general, not necessarily bicolored:

Definition 3.1 *A dag is a* 1-blocker *if for some color B, (i) $\Phi = |B^+| + |B^-|$; (ii) $B^+, \overline{B}^+ \neq \emptyset$; (iii) $R^-(B) \subseteq \overline{B}$.*

See Fig. 3. Recall the function R^- introduced at the start of Section 2. In a bicolored graph we can replace $\overline{B}$ by W in this definition. A 1-blocker always has $OPT > \Phi$. This fact is not needed in this section, and is proved in Lemma 4.3.

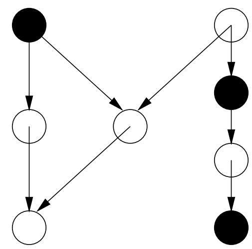


Figure 3: A 1-blocker.

Lemma 3.2 *A bicolored dag with $\Phi = |B^+| + |B^-| > |W^+| + |W^-|$ has $OPT = \Phi$ unless the dag is a 1-blocker.*

Proof: If $B^+ = \emptyset$ then $|B^-| = |B^+| + |B^-| = \Phi \geq |V^-|$ implies $V^- \subseteq B$. Hence the graph H is consistently-colored and Lemma 2.7 shows $OPT = \Phi$. Directional symmetry shows $OPT = \Phi$ if $B^- = \emptyset$. So assume $B^+, B^- \neq \emptyset$.

Assume H is not a 1-blocker. This implies $r(B)$. In proof, first suppose $W^+ = \emptyset$. Thus $V^+ \subseteq B$. Now $B^- \neq \emptyset$ gives the desired conclusion. So assume $W^+ \neq \emptyset$. Since H is not a 1-blocker, condition (iii) fails, which means $r(B)$.

We will maintain as *witnesses* two black vertices, $b^+ \in B^+$ and $b^- \in B^-$, where $r(b^+, b^-)$. Say that a pair of vertices x^-, y^+ is *witness-compatible* if it is compatible, and in addition for x^+ and y^- as in Definition 2.4, $x^- = b^-$ implies $y^- \in B$, and $y^+ = b^+$ implies $x^+ \in B$.

If there is a witness-compatible pair x^-, y^+ augment it. Let K be the resulting dag. The augment decreases Φ by 1 and K satisfies the hypothesis of the lemma, as in the proof of Lemma 2.9. K also has two witnesses. This is clear if neither b^+ nor b^- is involved in the augment. Suppose $x^- = b^-$. K has source b^+ and sink y^-, both black with $r(b^+, y^-)$. Thus y^- can be designated as the new witness b^-. Similarly if $y^+ = b^+$ then x^+ can be designated as the new witness b^+.

So we can assume H has no witness-compatible pairs. We claim that H is either unicolored or rooted. The claim, along with Lemmas 2.2 and 2.3, implies we are done.

We prove the claim by contradiction. Both sets $V^+ - b^+$ and $V^- - b^-$ are nonempty, since H is not rooted. These sets together contain a white extreme vertex, since not all extremes are black. They contain a black extreme vertex, by the lemma's hypothesis on Φ. All these facts imply that some source of $V^+ - b^+$ and some sink of $V^- - b^-$ have opposite colors. Without loss of generality assume there are vertices $y^+ \in W^+$ and $y^- \in B^- - b^-$.

If $r(y^+, y^-)$ then b^-, y^+ is witness-compatible. If $\neg r(y^+, y^-)$ then y^-, y^+ is witness-compatible. This is the desired contradiction. $\square$

Lemma 3.3 *A bicolored dag with $\Phi = |B^+| + |B^-| = |W^+| + |W^-|$ has $OPT = \Phi$ unless the dag is a 1-blocker.*

Proof: First observe that $\Phi = |V^+| = |V^-| = |B^+| + |B^-| = |W^+| + |W^-|$. In proof, the hypothesis on Φ implies $2\Phi \ge |V^+| + |V^-| = (|B^+| + |B^-|) + (|W^+| + |W^-|) = 2\Phi$. Thus equality holds throughout and $\Phi = |V^+| = |V^-|$.

If $B^+ = \emptyset$ then $\Phi = |V^-| = |B^-|$ so $V^- \subseteq B$. The graph H is consistently-colored and Lemma 2.7 shows $OPT = \Phi$. Using symmetry we can assume $B^+, W^+ \neq \emptyset$. Now if $R^-(B) \subseteq W$ then H is a 1-blocker. So using symmetry we can assume $r(B)$ and $r(W)$.

We will maintain as witnesses four vertices, $b^+ \in B^+$, $b^- \in B^-$ $w^+ \in W^+$ and $w^- \in W^-$, where $r(b^+, b^-)$ and $r(w^+, w^-)$. Say that a pair of vertices x^-, y^+ is *witness-compatible* if it is compatible and in addition for x^+ and y^- as in Definition 2.4, y^- has the same color as x^- if x^- is a witness, and x^+ has the same color as y^+ if y^+ is a witness.

If there is a witness-compatible pair x^-, y^+ augment it. This preserves the hypothesis of the lemma and the existence of our four witnesses. This is proved by the same argument as Lemma 3.2. Note this argument works even when x^- and y^+ are both witnesses.

So we can assume H has no witness-compatible pairs. We claim there are no extreme vertices other than the four witnesses. In proof, suppose without loss of generality there is another source $y^+ \in W^+ - w^+$. Since $\Phi = |V^+| = |V^-| = |B^+| + |B^-|$ there is a sink $y^- \in B^- - b^-$. If $r(y^+, y^-)$ then b^-, y^+ is witness-compatible, a contradiction. If $\neg r(y^+, y^-)$ then y^-, y^+ is witness-compatible, another contradiction.

The claim implies we are done, as follows: Augment b^-, w^+. This reduces Φ from 2 to 1, and Lemma 2.3 completes the argument. $\square$

Corollary 2.10 and Lemmas 3.2 and 3.3 give the following.

Lemma 3.4 *A bicolored dag has $OPT = \Phi$ unless it is a 1-blocker.* $\square$

We extend part of this analysis to arbitrarily colored dags as follows.

Lemma 3.5 *An arbitrarily colored dag H with $\Phi = |V^-| > |V^+|, |B^+| + |B^-|$ for every $B \in \mathcal{P}$ has $OPT = \Phi$.*

Proof: The lemma has been proved if H is rooted or bicolored (in Lemma 2.3 and Corollary 2.10) so assume the opposite. Lemma 2.8 shows there is a compatible pair. We will show how to augment a compatible pair so that Φ decreases by one, and the new graph either has $OPT = \Phi$ or continues to satisfy the inequality of the lemma. By induction this completes the proof. Call a color B *big* if $|B^+| + |B^-| = \Phi - 1$.

Suppose a given compatible pair contains a vertex from every big color. Augment it. This decreases $|V^-|$ and $|V^+|$ each by 1, so Φ decreases by 1. Furthermore the inequality of the lemma is preserved. This completes the analysis for this case. So from now on assume no compatible pair contains a vertex from every big color.

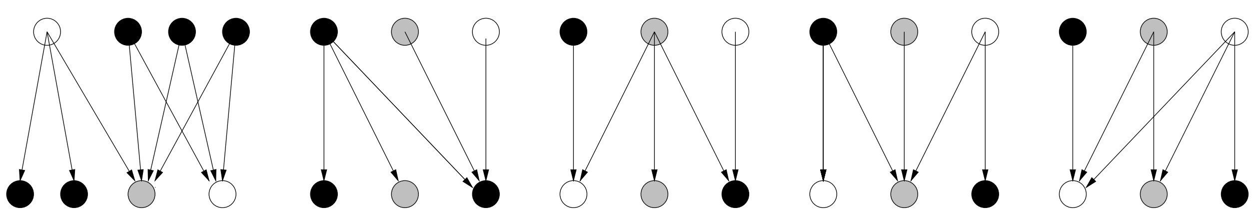
Suppose there are two (or more) big colors, say B and W. Since H has fewer than 2Φ extreme vertices and is not bicolored, exactly one extreme vertex g is not colored B or W. Let g be a source. (A symmetric argument holds if g is a sink.) Without loss of generality g precedes a black sink. Change the color of g to black. This does not change Φ. The resulting bicolored graph is not a 1-blocker since $r(B)$ and W has fewer than Φ extremes. Hence Lemma 3.4 gives the desired conclusion.

We can now assume there is a unique big color B and no compatible pair contains a vertex of B. Take a compatible pair x^-, y^+. Every vertex of B^+ is incompatible with x^- and so precedes x^-. Every vertex of B^- is incompatible with y^+ and so is preceded by y^+. B^+ and B^- are both nonempty. (This follows since $\Phi = |V^-| > |V^+| \geq 2$ implies B has at least two extremes. If all these extremes are sources (sinks) then some compatible pair involves B by Lemma 2.5.) Now augment the pair x^-, y^+. The augment decreases Φ by 1 and makes $\Phi = |B^+| + |B^-|$. Also the new graph K has $r(B)$. Take B and $\overline{B}$ as the two colors of K. $\overline{B}$ has less than Φ extreme vertices, since H has $2\Phi > |V^+| + |V^-|$. We conclude that K is bicolored and is not a 1-blocker. A solution using $OPT = \Phi$ new edges on K gives a similar solution on H. □

4 Blockers

We turn to general graphs. There are four types of blockers, illustrated in Fig. 4. For convenience we repeat Definition 3.1 for 1-blockers here.

Definition 4.1 *A graph is a* blocker *if it is a 1-,2-,3- or 4-blocker as defined by the conditions below. In these conditions B and W denote two distinct colors.*
A 1-blocker *has*
(i) $\Phi = |B^+| + |B^-|$;
(ii) $B^+, \overline{B}^+ \neq \emptyset$;
(iii) $R^-(B) \subseteq \overline{B}$.
The remaining blockers have the same first condition:
(i) $\Phi = |V^+| = |V^-|$.
A 2-blocker *has*
(ii) $B^+, \overline{B}^+ \neq \emptyset$;
(iii) $R^-(\overline{B}) \subseteq B$.
A 3-blocker *is a* 3^+*-blocker or a* 3^-*-blocker, where a* 3^+-blocker *has*
(ii) $B^+, W^+ \neq \emptyset$;
(iii) $R^-(B) \subseteq W$, $R^-(W) \subseteq B$, $|V^+ - (B \cup W)| = 1$;
and a 3^--blocker *satisfies the reverse conditions,*
(ii) $B^-, W^- \neq \emptyset$;
(iii) $R^+(B) \subseteq W$, $R^+(W) \subseteq B$, $|V^- - (B \cup W)| = 1$.
A 4-blocker *has*
(ii) $B^+, B^- \neq \emptyset$;
(iii) $R^-(B) \cup R^+(B) \subseteq W$, $|B^+| + |B^-| = \Phi - 1$.



(a) 1-blocker (b) 2-blocker (c) 3^+-blocker (d) 3^--blocker (e) 4-blocker

Figure 4: Simple examples of blockers.

The five varieties of blockers (differentiating between 3^+ and 3^-) are distinct. For instance each blocker in Fig. 4 satisfies only one set of blocker conditions.

The *reachability conditions* of a blocker are the containment relations in condition (iii). We specify the color classes associated with a blocker by saying a *1- or 2-blocker for B*, or a *3- or 4-blocker for B, W*. The roles of B and W are asymmetric in a 4-blocker for B, W. We say a *4-blocker for B* if we do not wish to refer to the second color W. We say a 3^+- *(3^--) blocker for z* if z is the (unique) vertex of $V^+ - (B \cup W)$ $(V^- - (B \cup W))$.

Lemma 4.2 *The reverse graph of a blocker is a blocker of the same type (1,2,3 or 4). Furthermore the reverse graph of a 3^+- (3^--) blocker is a 3^-- (3^+-) blocker.*

Proof: The lemma is obvious except for 1- and 2-blockers. For these we use two simple properties of the reachability functions: Consider any two sets of vertices X and Y in H. The set $R^+(X)$ $(R^-(X))$ in H is equal to the set $R^-(X)$ $(R^+(X))$ in the reverse graph. In H, $R^-(X) \subseteq Y$ if and only if $R^+(\overline{Y}) \subseteq \overline{X}$. Now we consider the two types of blockers.

1-blocker. The reachability condition $R^-(B) \subseteq \overline{B}$ is equivalent to $R^+(B) \subseteq \overline{B}$, which in the reverse graph is the reachability condition $R^-(B) \subseteq \overline{B}$. Next we verify condition (ii) in the reverse graph. Conditions (ii) and (iii) in H imply $\overline{B}^- \neq \emptyset$. Conditions (i) and (ii) imply $|B^+| + |B^-| = \Phi \geq |V^+| > |B^+|$, so $B^- \neq \emptyset$.

2-blocker. The argument is similar: The reachability condition $R^-(\overline{B}) \subseteq B$ is equivalent to $R^+(\overline{B}) \subseteq B$, which in the reverse graph is the reachability condition $R^-(\overline{B}) \subseteq B$. For condition (ii) in the reverse graph, conditions (ii) and (iii) in H imply $B^- \neq \emptyset$. Conditions (i) and (ii) imply $|V^-| = \Phi \geq |B^+| + |B^-| > |B^-|$, so $\overline{B}^- \neq \emptyset$. □

Lemma 4.3 *Any blocker has $OPT > \Phi$.*

Proof: We consider each of the four types of blockers in turn.

1-blocker. Let X be the set of all vertices preceded by a vertex of B^+. X is a nonempty proper subset of V, since $B^+ \subseteq X$ and $X \cap \overline{B}^+ = \emptyset$. Hence some new edge e leaves X. If $OPT = \Phi = |B^+| + |B^-|$ then every new edge either leaves a black sink or enters a black source, so e either leaves a black sink of X or enters a black source of $\overline{X}$. Such a sink does not exist since $R^-(B) \subseteq \overline{B}$, and such a source does not exist since $B^+ \subseteq X$. We conclude $OPT > \Phi$.

We give a similar argument for each of the remaining blockers. They all have $\Phi = |V^+| = |V^-|$. We will assume that $OPT = \Phi$. This implies every new edge must leave a sink and enter a source. For each blocker we will derive a contradiction.

2-blocker. Let X be the set of all vertices preceded by a vertex of $\overline{B}^+$. X is a nonempty proper subset of V, since $\overline{B}^+ \subseteq X$ and $X \cap B^+ = \emptyset$. Hence some new edge e leaves X, i.e., it leaves a sink of X and enters a source of $\overline{X}$. Every sink of X is black since $R^-(\overline{B}) \subseteq B$, and every source of $\overline{X}$ is black since $\overline{B}^+ \subseteq X$. Hence e joins two black vertices. This contradiction shows $OPT > \Phi$.

3-blocker. Let H be a 3^+-blocker for B, W and vertex z. Let X be the set of all vertices preceded by a vertex of B^+. X is a nonempty proper subset of V, since $B^+ \subseteq X$ and $z \notin X$. Hence some new edge e leaves X, i.e., it leaves a sink of X and enters a source of $\overline{X}$. The sink of X is white since $R^-(B) \subseteq W$. This implies the source of $\overline{X}$ is z, since $B^+ \subseteq X$. We conclude e is directed from a white sink to z. A similar argument shows that a new edge e' is directed from a black sink to z. Clearly $e \neq e'$. This gives the desired contradiction, since $OPT = \Phi$ implies at most one new edge is directed to z.

If H is a 3^--blocker the lemma follows from the previous argument with Lemma 4.2.

4-blocker. Let X be the set of all vertices preceded by a vertex of B^+. X is a nonempty proper subset of V, since $B^+ \subseteq X$ and $X \cap B^- = \emptyset$. (The latter holds since $R^-(B) \subseteq W$.) Hence a new edge e leaves X. The condition $R^-(B) \subseteq W$ implies e goes from a white sink to a non-black source.

The same argument on the reverse graph shows a new edge e' goes from a non-black sink to a white source. A new edge cannot join two white vertices so $e' \neq e$. Thus we have two new edges, neither incident to a black vertex. But $|B^+| + |B^-| = \Phi - 1 = OPT - 1$ implies there is at most one such new edge. $\square$

The next lemma is needed only for the general case of our theorem in Sections 6–7. Call a dag a *proper i-blocker* if it is an i-blocker and in addition for i equal to 3 or 4, it is not also a 1- or 2-blocker. Note that Lemma 4.2 implies the reverse of a proper i-blocker is a proper i-blocker.

Lemma 4.4 *A 3- or 4-blocker for B, W has $r(\overline{B \cup W})$ if it is proper.*

Proof: Consider a proper 3-blocker H. Lemma 4.2 implies we can assume H is a 3^+-blocker. We claim $\overline{B \cup W}$ contains a sink. This claim establishes the lemma for 3^+-blockers, since any source that precedes such a sink must be in $\overline{B \cup W}$ by the reachability conditions of a 3^+-blocker.

To prove the claim note that $\Phi > |B^+| + |B^-|, |W^+| + |W^-|$ since H is not a 1-blocker. Since a 3^+-blocker has 2Φ extremes, this implies at least two extremes are not in $B \cup W$. The definition of 3^+-blocker shows at least one sink is not in $B \cup W$.

Suppose H is a proper 4-blocker. Since H is not a 2-blocker and $B^+, W^+ \neq \emptyset$, $R^-(\overline{W}) \not\subseteq W$. So let x be a source of $\overline{W}$ that precedes a sink y of $\overline{W}$. The reachability conditions of a 4-blocker show $x, y \notin B$. Thus x and y are the desired vertices. $\square$

We conclude this section by analyzing another type of arbitrarily colored dag.

Lemma 4.5 *An arbitrarily colored dag H with $\Phi = |B^+| + |B^-|$ for some $B \in \mathcal{P}$ has $OPT = \Phi$ unless H is a 1- or 2-blocker.*

Proof: Let K be the given dag H with color classes taken to be B and $\overline{B}$. K has the same value of Φ as H, as shown at the end of Lemma 2.9. Obviously a solution on K is a valid solution on H.

Lemma 3.4 shows K has $OPT = \Phi$ unless K is a 1-blocker. If K is a 1-blocker for B then so is H. Suppose K is a 1-blocker for $\overline{B}$. Thus $\Phi = |\overline{B}^+| + |\overline{B}^-|$. Hence $|V^+| + |V^-| = |B^+| + |B^-| + |\overline{B}^+| + |\overline{B}^-| = 2\Phi \geq |V^+| + |V^-|$, implying $\Phi = |V^+| = |V^-|$. This, along with $R^-(\overline{B}) \subseteq B$, shows H is a 2-blocker for B. $\square$

Lemmas 4.5, 3.5 and symmetry leave only one case to complete the analysis of arbitrarily colored dags, $\Phi = |V^-| = |V^+| > |B^+| + |B^-|$ for every $B \in \mathcal{P}$. The next section analyzes this case for the dags that arise in bipartite strong-connectivity augmentation. Sections 6–7 do the same for general augmentation.

5 Bipartite augmentation theorem

Consider the bipartite strong-connectivity augmentation problem. The reduction of Section 2 transforms a given bipartite digraph into a dag H whose nonsingleton vertices have only two colors. We call such a dag *core-bicolored.* This section proves our theorem for core-bicolored dags.

This solves the cable-framework rigidity problem. A dag is *core-bicolored for B, W* if every vertex not colored B or W is a singleton.

Lemma 5.1 *A core-bicolored dag for B, W is consistently-colored if it is not a blocker but it satisfies conditions (i) and (iii) of a 1- or 2-blocker for B or a 3- or 4-blocker for B, W.*

Proof: Let H be a dag satisfying the hypotheses of the lemma. We consider each of the five varieties of blockers in turn. For each variety we assume H satisfies conditions (i) and (iii) of that blocker for B (or B, W) and we show H is consistently-colored.

1-blocker. If $\overline{B}^+ = \emptyset$ then $V^+ \subseteq B$ and condition (iii) implies $V^- = R^-(V) = R^-(B) \subseteq \overline{B}$. Thus H is consistently-colored. If $B^+ = \emptyset$ then $\Phi = |B^+| + |B^-| \geq |V^-|$ implies $V^- \subseteq B$. Again H is consistently-colored.

2-blocker. If $B^+ = \emptyset$ then $V^+ \subseteq \overline{B}$ and (iii) implies $V^- = R^-(V) = R^-(\overline{B}) \subseteq B$. Hence H is consistently-colored. If $\overline{B}^+ = \emptyset$ then $V^+ \subseteq B$. Thus $|B^+| + |B^-| \geq |V^+| = \Phi$ so equality holds, and $B^- = \emptyset$. Thus H is consistently-colored.

3^+-blocker. Without loss of generality assume $W^+ = \emptyset$. We can assume $B^+, B^- \neq \emptyset$, since otherwise H is consistently-colored. Since $|B^+| = |V^+| - 1$, (i) implies $\Phi = |B^+| + |B^-|$. Now H satisfies conditions (i) and (iii) of a 1-blocker for B, so the previous argument shows we are done.

3^--blocker. Assume $W^- = \emptyset$. The condition $R^+(B) \subseteq W$ implies $R^-(B) \subseteq \overline{B}$. Now the argument is the same as for 3^+-blockers.

4-blocker. Suppose $B^- = \emptyset$. Conditions (i) and (iii) imply there are $|V^+| - 1$ black sources. We can assume the remaining source is white, since otherwise H is consistently-colored. Now H satisfies (i) and (iii) of a 2-blocker for W, so the previous argument shows we are done.

Suppose $B^+ = \emptyset$. Conditions (i) and (iii) imply there are $|V^-| - 1$ black sinks. As above we can assume the remaining sink is white. The condition $R^+(B) \subseteq W$ implies $R^-(\overline{W}) \subseteq \overline{B}$, so $R^-(\overline{W}) \subseteq W$. So as above we are done. $\square$

Lemma 5.2 *If H is core-bicolored and $\Phi = |V^+| = |V^-| > |B^+| + |B^-|$ for every $B \in \mathcal{P}$ then $OPT = \Phi$ unless H is a blocker.*

Proof: Let H be core-bicolored for B, W. Let $G = V - B - W$, so every vertex of G is a singleton. Assume H is not a blocker. Lemma 2.7 shows we can assume H is not consistently-colored. Now Lemma 5.1 implies that H does not satisfy condition (iii) of a 2-,3- or 4-blocker for B (or B, W). Define the sets

$$\begin{aligned}\mathcal{W} &= \{(x^+, x^-) \in \overline{B}^+ \times \overline{B}^- : r(x^+, x^-)\};\\ \mathcal{B} &= \{(x^+, x^-) \in \overline{W}^+ \times \overline{W}^- : r(x^+, x^-)\}.\end{aligned}$$

We establish the properties of these sets in four claims.

Claim 1: $\mathcal{W}, \mathcal{B} \neq \emptyset$.
Proof: Condition (iii) of a 2-blocker for B fails. Hence $r(\overline{B})$, i.e., $\mathcal{W} \neq \emptyset$. ♠

Claim 2: No vertex is common to every ordered pair of $\mathcal{W} \cup \mathcal{B}$.

Proof: Suppose on the contrary that z is a common vertex. Clearly $z \in G$.

First assume $z \in G^+$. Then no vertex of B^+ precedes a vertex of $\overline{W}^-$, i.e., $R^-(B) \subseteq W$. Similarly $R^-(W) \subseteq B$. Also similarly, $G^+ = \{z\}$. We have proved condition (iii) of a 3^+-blocker for B, W, a contradiction.

A symmetric argument holds if $z \in G^-$: No vertex of B^- is preceded by a vertex of $\overline{W}^+$, i.e., $R^+(B) \subseteq W$. Similarly $R^+(W) \subseteq B$ and $G^- = \{z\}$. Thus condition (iii) of a 3^--blocker for B, W holds, a contradiction. ♠

Claim 3: $\mathcal{W} \cup \mathcal{B}$ contains two ordered pairs that do not contain a common vertex.

Proof: Take any pair $(x^+, x^-) \in \mathcal{W} \cup \mathcal{B}$. Since x^+ is not in every pair of $\mathcal{W} \cup \mathcal{B}$, there is a pair containing some $y^+ \in V^+ - x^+$. The pair must be (y^+, x^-), or else we are done. Similarly we have a pair (x^+, y^-) with $y^- \in V^- - x^-$. Now the desired pairs are (y^+, x^-) and (x^+, y^-). ♠

Claim 4: Some ordered pair in $\mathcal{W}$ and some ordered pair in $\mathcal{B}$ do not contain a common vertex.

Proof: Assume for the purpose of contradiction that Claim 4 fails. Claim 3 shows there are ordered pairs $(x^+, x^-), (y^+, y^-) \in \mathcal{B} \cup \mathcal{W}$ that have all four vertices distinct. Without loss of generality assume both pairs belong to $\mathcal{W} - \mathcal{B}$. Hence at least one vertex of each pair is white. Since Claim 4 fails, any pair of $\mathcal{B}$ has a vertex in common with (x^+, x^-) and a vertex in common with (y^+, y^-). Claim 1 shows there is such a pair, so let it be (x^+, y^-). This makes

$$x^+, y^- \in G,\ x^-, y^+ \in W.$$

It is now clear that $\mathcal{B} = \{(x^+, y^-)\}$. This implies $R^-(B) \cup R^+(B) \subseteq W$.

Next we show $G = \{x^+, y^-\}$. Suppose there is a vertex $g^+ \in G^+ - x^+$. Let g^+ precede a sink z^-. Since $(g^+, z^-) \notin \mathcal{B}$, $z^- \in W$. Now we have $z^- \neq y^-$, $(g^+, z^-) \in \mathcal{W}$ and $(x^+, y^-) \in \mathcal{B}$, contradicting the failure of Case 4. A symmetric argument applies to a vertex $g^- \in G^- - y^-$.

Since $|G^+| + |G^-| = 2$, the hypothesis of the lemma implies $|B^+| + |B^-| = |W^+| + |W^-| = |V^+| - 1$. Hence condition (iii) of a 4-blocker for B, W is satisfied, a contradiction. ♠

By Claim 4, take ordered pairs $(x^+, x^-) \in \mathcal{B}$ and $(y^+, y^-) \in \mathcal{W}$ with all four vertices distinct. Recolor the vertices of G to B or W so $x^+, x^- \in B$, $y^+, y^- \in W$ and if possible, $|V^+| \geq |B^+| + |B^-|, |W^+| + |W^-|$. If possible this recoloring gives a bicolored graph K that has the same value of Φ and is not a 1-blocker. Hence Lemma 3.4 shows K has $OPT = \Phi$. The solution to the augmentation problem on K is valid on H and shows H has $OPT = \Phi$.

Suppose the recoloring is impossible. The inequality hypothesized in the lemma shows that without loss of generality, $x^+, x^- \in G$ and $|V^+| - 1 = |B^+| + |B^-|$. Since condition (iii) of a 4-blocker for B, W does not hold, $R^-(B) \cup R^+(B) \not\subseteq W$. Thus $\mathcal{B}$ contains an ordered pair (z^+, z^-) with at least one black vertex.

Suppose the pairs $(z^+, z^-) \in \mathcal{B}$ and $(y^+, y^-) \in \mathcal{W}$ have all four vertices distinct. Use these pairs in the recoloring procedure given above. The procedure succeeds and shows $OPT = \Phi$. To prove this note the procedure fails only if $y^+, y^- \in G$ and $|V^+| - 1 = |W^+| + |W^-|$. But this implies the contradiction $2|V^+| = |V^+| + |V^-| \geq 2(|V^+| - 1) + 4$.

Finally suppose the vertices z^+, z^-, y^+, y^- are not all distinct. The unique common vertex is the vertex of $\{z^+, z^-\} \cap G$. This makes the four vertices z^+, z^-, x^+, x^- distinct. Furthermore $|W^+| + |W^-| \leq |V^+| - 2$, since $|B^+| + |B^-| = |V^+| - 1$ and $|G^+| + |G^-| \geq 3$. Thus we can use the pairs $(z^+, z^-) \in \mathcal{B}$ and $(x^+, x^-) \in \mathcal{W}$ in the recoloring procedure to show $OPT = \Phi$. □

Lemma 5.3 *Any core-bicolored dag has $OPT = \Phi$ unless it is a blocker, in which case $OPT = \Phi + 1$.*

Proof: If H is a blocker then $OPT = \Phi + 1$ by Lemmas 4.3 and 2.9. If H is not a blocker then $OPT = \Phi$ by Lemmas 4.5, 3.5 and 5.2.

Finally recall that we assumed the graph H has no isolated vertices. If there are isolated vertices, we use the transformation of Proposition 2.1. The transformed graph is a blocker if and only if the original graph is a blocker. This follows from the observation that the transformation does not change any of the colors appearing in any set $R^-(X)$, $R^+(X)$. □

6 Merging colors

This section and the next prove the general case of our theorem. We can assume $\mathcal{P}$ has at least three nonempty colors, else Lemma 3.4 applies. The main result of this section (Lemma 6.2) shows we can reduce $\mathcal{P}$ to exactly three classes. This is done by merging color classes. To *merge* color classes C_i, $i = 1, \ldots, k$ means to replace them by a new color $\bigcup_1^k C_i$ in $\mathcal{P}$. Throughout this section assume H is the given graph with partition $\mathcal{P}$, where $\mathcal{P}$ contains at least four nonempty color classes.

We start with two observations that follow easily from the definition of blocker. First, if H is not a blocker and a merge does not change Φ but results in a blocker for color B or colors B, W, then B or W is not a class of $\mathcal{P}$. Second, if a color class with no extreme vertices is merged with any other color, the new graph is an i-blocker if and only if the original graph is. In particular we can assume every color of $\mathcal{P}$ contains an extreme vertex.

Lemma 6.1 *Let H be a blocker with exactly three colors C_i, $i = 1, 2, 3$.*

(i) Suppose $r(C_i)$ for $i = 1, 2$. Then H is a 1- or 4-blocker for C_3.

(ii) Suppose $r(C_1)$, $r(C_2, C_3)$ and either $r(C_1, C_3)$ and $|C_1^+| > 1$ or $r(C_3, C_1)$ and $|C_1^-| > 1$. Then H is a 1-blocker for C_2 or C_3, or a 4-blocker for C_2.

Proof: First consider an arbitrary dag having a color C with $r(C)$. The definition of blocker shows that H is not a 1-, 3- or 4-blocker for C. If H is a 2-blocker it is a 2-blocker for C.

Part (i) follows by applying the above observation. We turn to part (ii). If H is a 2-blocker the above observation shows it is a 2-blocker for C_1, but $r(C_2, C_3)$ makes this is impossible. If H is a 3^+-blocker, say for vertex z, then $z \in C_1$ (since $r(C_1)$). But this implies $|C_1^+| = 1$ and $\neg r(C_3, C_1)$, contradicting the hypothesis. Similarly if H is a 3^--blocker for z then $z \in C_1$. But this implies $|C_1^-| = 1$ and $\neg r(C_1, C_3)$, contradicting the hypothesis. H is not a 4-blocker for C_1 by the preliminary observation. Suppose H is a 4-blocker for C_3. The other color of the blocker must be C_2 since $r(C_2, C_3)$, but it must be C_1 since $r(C_1, C_3)$ or $r(C_3, C_1)$. This contradiction gives part (ii). □

Lemma 6.2 *If a graph H has $|\mathcal{P}| \geq 4$ then some merge gives a new graph with the same value of Φ and $|\mathcal{P}| = 3$. Furthermore the new graph is a blocker only if H is.*

Proof: Call a merge of $\mathcal{P}$ into three classes *permissible* if it does not change Φ. A permissible merge can be found by repeatedly merging two color classes until a merge of classes C_1 and C_2 would produce a class with at least Φ extreme vertices. Then the desired permissible merge consists of classes C_1, C_2 and $\overline{C_1 \cup C_2}$.

For the rest of the argument assume H is not a blocker. We must show some permissible merge gives a graph that is not a blocker. Order the blocker types as 2,1,3,4. We will consider the blockers in this order. For each type i, we suppose K is an i-blocker that results from a permissible merge of H. We will construct a new permissible merge of H which results in a graph L. We will show that L is not a blocker of type i or a type listed before i. Repeated application of this argument eventually gives the desired merge.

2- and 1-blockers.

Claim: Let V be the disjoint union of X and Y, where X and Y are both the result of a merge. Suppose further that $R^-(X) \subseteq Y$, and $X^+, X^- \neq \emptyset$. Then X (Y) can be partitioned into disjoint sets X_i (Y_i), $i = 0, 1$, where each X_i (Y_i) is the result of merging classes of $\mathcal{P}$ contained in X (Y), such that $r(X_i \cup Y_i)$ for $i = 0, 1$.
Proof: Start by choosing $X_0 \subseteq X$ as a class of $\mathcal{P}$ containing a source. Take $Y_0 \subseteq Y$ as a class of $\mathcal{P}$ containing a sink preceded by X_0^+. Let X_1 and Y_1 consist of the remaining classes of $\mathcal{P}$ contained in X (Y). If $r(X_1 \cup Y_1)$ we are done. Hence assume the opposite:

$$R^-(X_1) \subseteq Y_0 \text{ and } R^-(Y_1) \subseteq X_0 \cup Y_0.$$

We will prove $r(X_0 \cup Y_1)$ and $r(X_1 \cup Y_0)$. Clearly this implies the Claim.

The first relation $r(X_0 \cup Y_1)$ holds if $r(Y_1, X_0)$ so suppose the opposite. This implies $R^-(Y_1) \subseteq Y_0$. Next note that $Y_0^+ \neq \emptyset$. This follows since $X^- \neq \emptyset$, so a sink in X is preceded by a source of Y, and that source cannot be in Y_1. Now since H is not a 2-blocker for Y_0, $R^-(X_0) \not\subseteq Y_0$. Thus $r(X_0, Y_1)$. This implies the first relation.

Consider the second relation $r(X_1 \cup Y_0)$. If $X_1^+ \neq \emptyset$ then $r(X_1, Y_0)$. If $X_1^+ = \emptyset$ then $X_1^- \neq \emptyset$, which implies $r(Y_0, X_1)$. ♠

Now suppose K is a 2- or 1-blocker. First note that the Claim applies to B and $\overline{B}$. (For a 2-blocker the Claim requires $\overline{B}^- \neq \emptyset$. For a 1-blocker the Claim requires $B^- \neq \emptyset$. Both of these are shown in the proof of Lemma 4.2.)

So let B ($\overline{B}$) be partitioned into sets A_i (C_i), $i = 0, 1$, where $r(A_i \cup C_i)$ for $i = 0, 1$. For at least one index i, $A_i \cup C_i$ contains at most Φ extreme vertices. Without loss of generality let the index be $i = 0$. Let the three new color classes be $A_0 \cup C_0$, A_1 and C_1.

The new graph L is not a 2-blocker. In proof $r(A_0 \cup C_0)$ implies L can only be a 2-blocker for $X = A_0 \cup C_0$. But this is inconsistent with $r(\overline{X})$, i.e., $r(A_1 \cup C_1)$.

L is not a 1-blocker if K is a 1-blocker. In proof suppose L is a 1-blocker for color X. $X \neq A_0 \cup C_0$ because $r(A_0 \cup C_0)$. $X \neq A_1, C_1$ since $A_0 \cup A_1$ contains exactly Φ extreme vertices, so $C_0 \cup C_1$ contains at most Φ extreme vertices. This implies A_1 and C_1 both contain fewer than Φ extremes.

3^+*-blockers.* Suppose K is a proper 3^+-blocker for B, W.

Claim: Without loss of generality B can be partitioned into disjoint sets A_i, $i = 0, 1$, where each A_i is a merge of classes of $\mathcal{P}$, such that (i) $r(A_0, W)$; (ii) $r(A_1, W)$ or $r(W, A_1)$.
Proof: Suppose B contains at least two classes of $\mathcal{P}$. Let A_0 be a class of $\mathcal{P}$ contained in B that contains a source, and let A_1 consist of the remaining classes of $\mathcal{P}$ in B. Clearly property (i) holds. Suppose (ii) fails. Thus $A_1^+ = \emptyset$ and $\neg r(W, A_1)$. This means that K is a proper 3^+-blocker for A_0, W. W must contain at least two classes of $\mathcal{P}$ since H is not a blocker. Repeat the above argument on the set W instead of B. This time (ii) cannot fail, since that would make H a 3-blocker. ♠

Assume B, A_0 and A_1 satisfy the Claim. Let $G = \overline{B \cup W}$. Either $A_0 \cup W$ or $A_1 \cup G$ contains at most Φ extreme vertices. Thus we get a permissible merging consisting of either colors $A_0 \cup W$, A_1 and G or A_0, W and $A_1 \cup G$. Recall the relation $r(G)$ from Lemma 4.4. Also note that A_0, A_1 and W each have fewer than Φ extreme vertices, since both B and W contain fewer than Φ extremes in any proper 3-blocker for B, W.

Suppose the permissible merging is $A_0 \cup W$, A_1 and G. Lemma 6.1(i) with $r(A_0 \cup W)$ and $r(G)$ shows that if L is a blocker it is a 1-blocker for A_1 or a 4-blocker. The former is impossible since as noted, A_1 has fewer than Φ extreme vertices.

Suppose the permissible merging is $A_1 \cup G$, A_0 and W. These sets satisfy the hypothesis of Lemma 6.1(ii), since we have $r(A_1 \cup G)$, $r(A_0, W)$ and furthermore, either $r(A_1, W)$, in which case $A_1 \cup G$ has at least two sources, or $r(W, A_1)$, in which case $A_1 \cup G$ has at least two sinks. Now Lemma 6.1(ii) shows that if L is a blocker it is either a 1-blocker for A_0 or W, or a 4-blocker. It is not a 1-blocker because as noted above, both A_0 and W have fewer than Φ extreme vertices.

3^-*-blockers.* Suppose K is a proper 3^--blocker. Lemma 4.2 shows its reverse graph is a proper 3^+-blocker. Apply the construction for proper 3^+-blockers to the reverse graph. If the new merging is a blocker, it must be a 4-blocker. The same holds for the original graph.

4-blockers. Suppose K is a proper 4-blocker. Let $X = B$, $Y = W$ if B contains two or more classes of $\mathcal{P}$; otherwise let $X = W$, $Y = B$. So X contains two or more classes of $\mathcal{P}$. Let $G = \overline{B \cup W} = \overline{X \cup Y}$.

Claim: X can be partitioned into disjoint sets A_i, $i = 0, 1$, where each A_i is a merge of classes of $\mathcal{P}$, such that $r(A_0, Y)$ and either $r(A_1, Y)$ or $r(Y, A_1)$. In addition A_0 and A_1 each contain fewer than $\Phi - 1$ extreme vertices. Furthermore $r(G)$.
Proof: The last part of the Claim $r(G)$ holds by Lemma 4.4.

Suppose $X = B$. Let A_0 be a class of $\mathcal{P}$ contained in X that contains a source, and let A_1 consist of the remaining classes in B. Hence $r(A_0, W)$. Furthermore $r(A_1, W)$ or $r(W, A_1)$. In addition each A_i has fewer than $\Phi - 1$ extreme vertices since $A_i \subset B$.

Suppose $X = W$. Let A_0 be a class of $\mathcal{P}$ contained in W such that $r(A_0, B)$. Let A_1 consist of the remaining classes of $\mathcal{P}$ in W. We have $r(A_1, B)$ or $r(B, A_1)$, since otherwise H is a 4-blocker for B, A_0 (both classes of $\mathcal{P}$). Each set A_i contains fewer than $2\Phi - (\Phi - 1) - 2 = \Phi - 1$ extreme vertices. ♠

Either $A_0 \cup Y$ or $A_1 \cup G$ contains at most Φ extreme vertices. Thus we get a permissible merging consisting of colors $A_0 \cup Y$, A_1 and G or A_0, Y and $A_1 \cup G$.

Consider a permissible merging $A_0 \cup Y$, G and A_1. These sets satisfy the hypothesis of Lemma 6.1(i). The new graph L is not a 1- or 4-blocker for A_1, since the Claim shows A_1 has too few extreme vertices. Thus L is not a blocker.

Consider a permissible merging $A_1 \cup G$, A_0 and Y. These sets satisfy the hypothesis of Lemma 6.1(ii). L is not a 1- or 4-blocker for A_0, since the Claim shows A_0 has too few extremes. Finally L is not a 1-blocker for Y, since Y is B or W and K is a proper 4-blocker. □

We finish this section with a simple result also proved by merging colors. The result can be seen from Lemma 5.3 but the following proof is independent of Section 5.

Lemma 6.3 *Let H be a tricolored dag with $\Phi = |V^+| = |V^-|$, where some color does not contain a source. Then $OPT = \Phi$ if H is not a blocker.*

Proof: Let the extreme vertex colors of H be B, W and G where $G^+ = \emptyset$. By Lemma 4.5 assume each color has fewer than Φ extreme vertices.

We can change the color of every extreme vertex of G to B or W without changing Φ. This follows since $2\Phi = |V^+| + |V^-|$, and we start with both B and W containing fewer than Φ extremes. In fact we can do this recoloring even with the requirement that a given vertex of G be colored B and another given vertex of G be colored W.

If every source has the same color then $\Phi = |V^+|$ implies H is consistently-colored. In this case $OPT = \Phi$ (Lemma 2.7). So assume $B^+, W^+ \neq \emptyset$. We can assume H is not a 2-blocker, so $r(\overline{B})$. This implies $r(W, \overline{B})$. Similarly $r(B, \overline{W})$.

G contains at least two sinks, since $|V^+| + |V^-| = 2\Phi \geq |B^+| + |B^-| + |W^+| + |W^-| + 2$. Now it is easy to see that there is a sink $w \in \overline{B}$ preceded by a source of W and a different sink $b \in \overline{W}$ preceded by a source of B.

If b is colored G change it to B. If w is colored G change it to W. Change the color of the remaining extreme vertices of G to B or W without changing Φ. The new bicolored graph is not a 1-blocker since $r(B)$ and $r(W)$. Lemma 3.4 shows it has a solution with $OPT = \Phi$ new edges, which gives a similar solution on H. $\square$

7 General augmentation theorem

This section solves the augmentation problem for tricolored dags. As mentioned at the end of Section 4 we need only consider the case $\Phi = |V^-| = |V^+| > |B^+| + |B^-|$ for every $B \in \mathcal{P}$.

We use the following notational conventions in this section. The three colors are designated as B, W and G ("black," "white" and "gray"). Sources of B, W, G, $\overline{B}, \overline{W}, \overline{G}$ are usually designated by b, w, g, $\overline{b}, \overline{w}, \overline{g}$ respectively. Sinks of B, W, G, $\overline{B}, \overline{W}, \overline{G}$ are usually designated by β, ω, γ, $\overline{\beta}, \overline{\omega}, \overline{\gamma}$ respectively.

We begin with several witness relations that are used in the proofs.

Lemma 7.1 *Let H be a tricolored blocker with colors B, W, G.*

(i) $r(B)$ and $r(\overline{B})$ imply H is not a 2-blocker.

(ii) $r(\overline{B})$, $r(\overline{W})$ and $r(\overline{G})$ imply H is not a 2-blocker.

(iii) $r(B, W)$, $r(W, G)$ and $r(G, B)$ imply H is a 1-blocker.

(iv) Suppose

$$r(\overline{b}, \omega),\ r(w, \beta),\ \text{and } r(g, \gamma)$$

for $\overline{b} \in \overline{B}^+$, $\omega \in W^-$, $w \in W^+$, $\beta \in B^-$, $g \in G^+$, $\gamma \in G^-$, where the six vertices are distinct (i.e., $\overline{b} \neq w, g$). Then either H is a 1-blocker or H is a 4-blocker for B.

Proof: Parts (i) and (ii) follow from the definition of 2-blocker.

(iii) H is not a 2-blocker for B because $r(W, G)$. H is not a 3^+-blocker for B because $r(B, W)$ and $r(W, G)$. Similarly H is not a 3^--blocker for B because $r(G, B)$ and $r(W, G)$. H is not a 4-blocker for B because $r(B, W)$ and $r(G, B)$. Now symmetry of the colors gives the desired conclusion.

(iv) $r(g,\gamma)$ and $r(w,\beta)$ show H is not a 2-blocker. If H is a 3^+-blocker then $r(g,\gamma)$ shows it is a 3^+-blocker for vertex g. Since $\overline{b} \neq g$ we must have $\overline{b} \in W$, but then $r(\overline{b},\omega)$ shows H is not a 3^+-blocker. If H is a 3^--blocker then $r(g,\gamma)$ shows it is a 3^--blocker for γ. But this is inconsistent with $r(\overline{b},\omega)$. $r(g,\gamma)$ shows H is not a 4-blocker for G. $r(w,\beta)$ and $r(\overline{b},\omega)$ show it is not a 4-blocker for W. $\square$

Corollary 7.2 *Suppose H is not a blocker.*

(*i*) *Suppose*

$$r(b,\chi),\ r(x,\beta),\ \textit{and}\ r(\overline{b},\overline{\beta})$$

for six distinct vertices $b \in B^+$, $\chi \in V^-$, $x \in V^+$, $\beta \in B^-$, $\overline{b} \in \overline{B}^+$, $\overline{\beta} \in \overline{B}^-$ where χ and x have different colors. Augmenting the compatible pair χ, x does not give a 2-blocker.

(*ii*) *Suppose*

$$r(g,\gamma),\ r(w,\chi),\ \textit{and}\ r(x,\omega)$$

for six distinct vertices $g \in G^+$, $\gamma \in G^-$, $w \in W^+$, $\chi \in V^-$, $x \in V^+$ and $\omega \in W^-$, where χ and x have different colors. Augmenting the compatible pair χ, x gives either a non-blocker, a 1-blocker, or a 4-blocker for B. The latter cannot occur if $B - \{x,\chi\}$ has less than $\Phi - 2$ extremes.

(*iii*) *Suppose*

$$r(b,\omega),\ r(w,\beta),\ r(g,\gamma),\ \text{and}\ r(\overline{b},\overline{\beta})$$

for eight distinct vertices $b \in B^+$, $\omega \in W^-$, $w \in W^+$, $\beta \in B^-$, $g \in G^+$, $\gamma \in G^-$, $\overline{b} \in \overline{B}^+$, $\overline{\beta} \in \overline{B}^-$ (i.e., $\overline{b} \neq w, g$ and $\overline{\beta} \neq \omega, \gamma$). Augmenting the compatible pair $\overline{\beta}, b$ gives either a non-blocker, a 1-blocker, or a 4-blocker for B. The latter cannot occur if B has less than $\Phi - 1$ extremes.

Proof:
(i) The new graph has $r(b,\beta)$ and $r(\overline{b},\overline{\beta})$, so it is not a 2-blocker by Lemma 7.1(i).
(ii) Follows from Lemma 6.1(*i*).
(iii) Follows from Lemma 7.1(iv). $\square$

Assumption.
For the rest of this section assume H is a dag with

$$\Phi = |V^-| = |V^+| > |B^+| + |B^-| \text{ for every } B \in \mathcal{P}.$$

Furthermore assume H has exactly three nonempty colors, each of which contains both a source and a sink (by Lemmas 6.2 and 6.3). Finally assume H is not a blocker. We will prove $OPT = \Phi$.

Lemma 2.8 shows H has a compatible pair. Augment such a pair. This decreases Φ by 1, by the assumption on Φ. The new graph K is still tricolored. If K is not a blocker induction (and the analysis of the other cases) shows we are done. So assume K is a blocker. It suffices to show how to augment a different compatible pair so the new graph is not a blocker.

We do this in a manner similar to Section 6: Order the blocker types as 2,3,4,1. For each type i, we suppose K is an i-blocker resulting from an augment of H. We will augment a new compatible pair of H to get a new graph L. We will show L is not a blocker of type i or a type listed before i. Repeated application of this argument eventually gives the desired augment. It is convenient to consider 1-blockers first since that case gets used by the others.

Our arguments involve the three graphs H, K, L. The functions R^- and R^+ refer to H, as does the predicate r. r_K and r_L refer to graphs K and L respectively.

The following simple principle is useful.

Lemma 7.3 *Suppose graph K results from augmenting the pair χ, x. Then $r_K(a,\delta)$ implies $r(a,\delta)$ when*

$$r(u,\chi),\ r(x,\xi),\ \text{and either } \neg r_K(a,\xi) \text{ or } \neg r_K(u,\delta)$$

for vertices $u, x, a \in V^+$, $\chi, \xi, \delta \in V^-$.

Proof: $r_K(a,\delta)$ with $\neg r(a,\delta)$ means the path from a to δ in K includes edge (χ, x). Thus $r_K(a,\xi)$ and $r_K(u,\delta)$. □

1-blockers

This section shows that if an augment gives a 1-blocker, a different augment gives a bicolored graph. In fact we prove a stronger result that is useful later on.

Suppose an augment of H gives a 1-blocker for B. B must have $\Phi - 1$ extremes (by the assumption on Φ) and the augment cannot involve a vertex colored B. The latter implies $\neg r(B)$. These facts show the following lemma handles the case of augments that give 1-blockers.

Lemma 7.4 *Let H be a dag having a color B with exactly $\Phi - 1$ extreme vertices and $\neg r(B)$. Then some augment of H gives a dag L that is not a blocker. Furthermore two colors of L can be merged without changing Φ to get a bicolored dag that is not a blocker.*

Proof: Vertices x^-, y^+ form a *good pair* if neither vertex is colored B, the vertices are compatible and augmenting x^-, y^+ gives a new graph L with

$$r_L(B) \text{ and } r_L(\overline{B}).$$

We will find a good pair of vertices. This suffices to prove the lemma: L has $\Phi = |B^+| + |B^-| = |\overline{B}^+| + |\overline{B}^-|$, so using colors $B, \overline{B}$ gives a bicolored graph that is not a 1-blocker, as desired. We will argue by contradiction. So suppose no good pair exists.

We first show there are four distinct vertices $b \in B^+$, $\omega \in W^-$, $g \in G^+$, $\beta \in B^-$ such that

$$r(b,\omega) \text{ and } r(g,\beta).$$

In proof $\neg r(B)$ implies there are vertices $b \in B^+$ and $\beta \in B^-$ with $r(b,\chi)$, $r(x,\beta)$ for some $x, \chi \notin B$. Since H is not a 4-blocker x and χ can be chosen to have different colors. Naming colors appropriately gives the desired vertices.

The compatible pair ω, g is a good pair if $r_L(\overline{B})$. (Whenever we refer to a good pair, L denotes the graph that results from augmenting the pair.) The latter holds if either $r(\overline{B} - g, \overline{B} - \omega)$ or $r(\overline{B} - g, \omega)$ and $r(g, \overline{B} - \omega)$. So neither of these two alternatives holds. The failure of the second alternative implies we can assume $\neg r(\overline{B}-g,\omega)$ (since we can replace H by its reverse graph). This plus the failure of the first alternative implies

$$R^{-}(\overline{B}-g)\subseteq B. \tag{1}$$

(1) with $r(\overline{B})$ (since H is not a 2-blocker) implies there is a vertex $\overline{\beta}\in\overline{B}^{-}$ with

$$r(g,\overline{\beta}).$$

(It is possible that $\overline{\beta}=\omega$.)

Next we show

$$r(B,G)\Longrightarrow G^{-}=\{\overline{\beta}\}\text{ and }R^{+}(W)\subseteq B. \tag{2}$$

Suppose $r(b',\gamma)$ for vertices $b'\in B^{+},\gamma\in G^{-}$. $W^{+}\neq\emptyset$ and (1) show there are vertices $w\in W^{+},\beta'\in B^{-}$ with $r(w,\beta')$. The compatible pair γ,w is good if $r_L(\overline{B})$, so $\neg r_L(\overline{B})$. Since $r(g,\overline{\beta})$ and $g\neq w$ we must have $\gamma=\overline{\beta}$. Thus $\overline{\beta}\in G$. Furthermore $R^{+}(W)\subseteq B$, since otherwise we can choose $\overline{\beta}\in W$.

We complete the proof of (2) by showing $G^{-}=\{\overline{\beta}\}$: Consider a vertex $\gamma'\in G^{-}-\overline{\beta}$. We have shown $\gamma=\overline{\beta}$ holds for every choice of γ and $\overline{\beta}$, so these two vertices are unique. We have $\neg r(B,\gamma')$ (since otherwise γ' could be chosen as γ), $\neg r(g,\gamma')$ (since otherwise γ' could be chosen as $\overline{\beta}$) and $\neg r(\overline{B}-g,\gamma')$ (by (1)). Hence γ' does not exist.

We can now assume without loss of generality that $\overline{\beta}\in G$: This relation has just been proved if $r(B,G)$. On the other hand $\neg r(B,G)$ implies any vertex of G^{-} can be chosen as $\overline{\beta}$.

Next we show $G^{+}=\{g\}$: Suppose $g'\in G^{+}-g$. g' precedes a black sink β' by (1). The pair ω,g' is compatible since we have $r(b,\omega)$ and $r(g',\beta')$. In fact ω,g' is a good pair: The relation $r_L(\overline{B})$ holds since $r(g,\overline{\beta})$ and $\overline{\beta}\neq\omega$ (because $\overline{\beta}\in G$). Since there are no good pairs, g' does not exist.

Since H is not a 3^{+}-blocker but $|G^{+}|=1$ and $R^{-}(W)\subseteq B$, we have $r(B,\overline{W})$. Together with the hypothesis $\neg r(B)$ we get $r(B,G)$. Hence the hypothesis and conclusion of (2) hold. Thus $|G^{+}|=|G^{-}|=1$, so W (as well as B) has exactly $\Phi-1$ extreme vertices. This along with $R^{-}(W)\cup R^{+}(W)\subseteq B$ makes H a 4-blocker, a contradiction. □

2-blockers

Lemma 7.5 *If an augment of H gives a graph K that is a 2-blocker, a different augment of H gives a graph L that is not a 2-blocker.*

Proof: Suppose K is a 2-blocker for B. Since H is not a 2-blocker there are vertices $x\in\overline{B}^{+},\chi\in\overline{B}^{-}$ with $r(x,\chi)$ and either x or χ (or both) a vertex in the augment. By possibly changing to the reverse graph (using Lemma 4.2) we can assume x is a vertex in the augment. Without loss of generality assume $x\in G^{+}$. We consider two cases.

Case 1. χ is not a vertex in the augment.

In this case the augment forming K gives four distinct vertices $b\in B^{+}$, $\overline{\gamma}\in\overline{G}^{-}$, $g\in G^{+}$, $\overline{\beta}\in\overline{B}^{-}$ such that

$$r(b,\overline{\gamma})\text{ and }r(g,\overline{\beta}).$$

Here we have replaced x and χ by g and $\overline{\beta}$ respectively, and b and $\overline{\gamma}$ are the other two vertices involved in the augment. $\overline{\gamma}$ must be in $\overline{G}$ since the edge added by the augment joins differently colored vertices. b must be in B because it precedes $\overline{\beta}$ in K, which is a 2-blocker for B. Finally $\overline{\gamma} \neq \overline{\beta}$ by the assumption of Case 1.

Next we show there are vertices $\overline{b} \in \overline{B}^+$, $\beta \in B^-$ that are distinct from the other four vertices such that

$$r(\overline{b}, \beta).$$

In proof, such a pair of vertices with $r_K(\overline{b}, \beta)$ exists, since a 2-blocker has $\overline{B}^+ \neq \emptyset$. The relation $r(\overline{b}, \beta)$ follows from Lemma 7.3 (using $\neg r_K(\overline{b}, \overline{\beta})$).

If $\overline{b}$ can be chosen to have a color different from $\overline{\gamma}$ then Corollary 7.2(i) applies to the six vertices $b, \overline{\gamma}, \overline{b}, \beta, g, \overline{\beta}$ and we are done. So suppose no such vertex $\overline{b}$ exists. This implies

$$\overline{\gamma},\ \overline{b} \in W.$$

In proof, the two vertices have different colors if $\overline{\gamma} \in B$. Hence $\overline{\gamma} \in W$ and so $\overline{b} \in W$.

We now show $R^-(W \cup G - g) \subseteq B$. If not then $r(W \cup G - g, \overline{B})$. This continues to hold in graph K, even if the only sink of $\overline{B}$ reachable from $W \cup G - g$ is $\overline{\gamma}$. But this contradicts K a 2-blocker.

The inclusion $R^-(W \cup G - g) \subseteq B$ implies $|G^+| = 1$ (otherwise a vertex of $G^+ - g$ could be chosen as $\overline{b}$; here we use $\overline{\gamma} \in W$). The inclusion also implies $R^-(W) \subseteq B$. Since H is not a 3^+-blocker we conclude $r(B, \overline{W})$. Hence we can take vertices $b' \in B^+$, $\overline{\omega} \in \overline{W}^-$ with

$$r(b', \overline{\omega}).$$

Corollary 7.2(i) applies to the six vertices $b', \overline{\omega}, \overline{b}, \beta, g, \overline{\beta}$ if the three sinks are distinct (recall $\overline{b} \in W$). That leaves us with two possibilities, $\overline{\omega} = \beta$ and $\overline{\omega} = \overline{\beta}$.

Suppose $\overline{\omega} = \beta$. Form L by augmenting the pair $\overline{\beta}, b$. If $b' \neq b$ then $r_L(b', \beta)$ and $r_L(g, \overline{\gamma})$, with $\overline{\gamma} \in W$, show L is not a 2-blocker by Lemma 7.1(i). If $b' = b$ then $r_L(g, \overline{\gamma})$, $r_L(g, \beta)$ and $r_L(\overline{b}, \beta)$ with $\overline{\gamma}, \overline{b} \in W$, show L is not a 2-blocker by Lemma 7.1(ii).

Suppose $\overline{\omega} = \overline{\beta}$. Form L by augmenting the pair β, g. We have $r_L(b, \overline{\gamma})$, $r_L(b', \overline{\beta})$ and $r_L(\overline{b}, \overline{\beta})$. The supposition implies $\overline{\beta} \in G$, and we have $\overline{\gamma}, \overline{b} \in W$. So Lemma 7.1(ii) shows L is not a 2-blocker.

Case 2. χ is a vertex in the augment.

The augment forming K gives four distinct vertices $b \in B^+$, $\omega \in W^-$, $g \in G^+$, $\beta \in B^-$ such that

$$r(b, \omega),\ r(g, \omega) \text{ and } r(g, \beta).$$

Here we have replaced x and χ by g and ω respectively. We can assume $b, \beta \in B$ because if not renaming vertices puts us in Case 1.

Lemma 6.3 shows there are vertices $w \in W^+$ and $\gamma \in G^-$. Clearly these vertices are distinct from the four vertices above. Since K is a 2-blocker, $\neg r_L(w, \gamma)$, so $\neg r(w, \gamma)$. Thus w, γ is a compatible pair. Form L by augmenting w, γ. The three relations displayed above hold in L. So Lemma 7.1(ii) shows L is not a 2-blocker. $\square$

3-blockers

Lemma 7.6 *If an augment of H gives a graph K that is a proper 3^+-blocker, a different augment of H gives a graph L that is not a 2- or 3-blocker.*

Proof: Suppose K is a 3^+-blocker for B, W and vertex $g \in G$. H has $|G^+|$ equal to 1 or 2. We consider these cases separately.

Case 1. $|G^+| = 2$.

Case 1.1. $R^-(B) \subseteq W$ and $R^-(W) \subseteq B$.

If B has $\Phi - 1$ extreme vertices we are done by Lemma 7.4. Hence we can assume both B and W have less than $\Phi - 1$ extremes. This implies $|G^-| \geq 2$.

We claim there are vertices $b \in B^+$, $\omega \in W^-$, $w \in W^+$, $\beta \in B^-$, $g \in G^+$ and $\gamma \in G^-$ with

$$r(b, \omega),\ r(w, \beta) \text{ and } r(g, \gamma).$$

The first two relations follow from the assumptions of Case 1.1. $r_K(G)$ holds since K is a proper 3-blocker (recall Lemma 4.4). This implies $r(G)$, since the augment involves a source of G^+. We have proved the last relation $r(g, \gamma)$.

There are vertices $g' \in G^+ - g$ and $\gamma' \in G^- - \gamma$. Suppose these vertices can be chosen so $r(g', \gamma')$. This gives eight vertices satisfying the hypothesis of Corollary 7.2(iii), so we are done. Now assume such vertices do not exist.

Case 1.1 implies $R^+(G) \subseteq G$. So the assumption just made implies $R^+(G) = \{g\}$. Thus g' precedes a sink of $B \cup W$. Without loss of generality $r(g', \omega')$ for some $\omega' \in W^-$. Now the relations $r(g, \gamma)$, $r(w, \beta)$ and $r(g', \omega')$ satisfy the hypothesis of Corollary 7.2(ii), so we are done.

If Case 1.1 does not hold then without loss of generality the following applies.

Case 1.2. $R^-(B) \not\subseteq W$.

The augment that produces K involves the source $g' \in G^+$. Hence there are vertices $b \in B^+$, $\beta \in B^-$, $\omega \in W^-$ with

$$r(b, \beta) \text{ and } r(g', \omega).$$

Here the augment is for β, g'. The assumption of Case 1.2 implies we must have β a nonwhite vertex preceded by a black source b. Since β is also non-gray, $\beta \in B$. Finally $\omega \in W$ since $r_K(b, \omega)$.

Since K is a proper 3-blocker there are vertices $g \in G^+ - g'$, $\gamma \in G^-$ with $r_K(g, \gamma)$. This implies $r(g, \gamma)$ by Lemma 7.3 (since $\neg r_K(b, \gamma)$). Similarly there are vertices $w \in W^+, \beta' \in B^- - \beta$ with $r(w, \beta')$ (use $\neg r_K(b, \beta')$). Now the relations $r(g, \gamma)$, $r(w, \beta')$ and $r(g', \omega)$ show Corollary 7.2(ii) applies and we are done.

Case 2. $|G^+| = 1$.

Case 2.1. $R^-(B) \not\subseteq W$, $R^-(W) \not\subseteq B$.

We can assume there are vertices $b \in B^+$, $\varpi \in \overline{W}^-$, $w \in W^+$, $\omega \in W^-$ such that

$$r(b, \varpi) \text{ and } r(w, \omega).$$

Here the augment is for ϖ, w. The source of the augment is not in G by Case 2, so without loss of generality $w \in W$. The first relation of Case 2.1 now shows $\varpi \in \overline{W}$ and is preceded by a source $b \in B$. Finally $\omega \in W^-$ since $r_K(b, \omega)$.

Since K is a 3^+-blocker it contains vertices $w' \in W^+ - w$, $\beta \in B^- - \overline{\omega}$ such that $r_K(w', \beta)$. Lemma 7.3 shows $r(w', \beta)$, since $\neg r_K(w', \omega)$. Now the relations $r(w, \omega)$, $r(b, \overline{\omega})$ and $r(w', \beta)$ show Corollary 7.2(ii) applies and we are done.

If Case 2.1 does not hold then without loss of generality the following applies.

Case 2.2. $R^-(B) \subseteq W$, $R^-(W) \not\subseteq B$.

Call vertices $w \in W^+$, $\overline{\beta} \in \overline{B}^-$ a *bad pair* if $r(w, \overline{\beta})$. The augment producing K involves a vertex in a bad pair. We examine three corresponding possibilities.

First suppose there is only one bad pair, and the augment involves both its vertices. Then there are vertices $g \in G^+$, $\gamma \in G^-$, $w \in W^+$, $\beta \in B^-$ such that

$$r(g, \gamma),\ r(w, \gamma) \text{ and } r(w, \beta),$$

where the augment is for the bad pair γ, w. $\beta \in B$ since w, β is not a bad pair. $\gamma \in G$ because $\gamma \in \overline{B}$ by assumption and $\gamma \in \overline{W}$ by the augment. This implies $g \in G$ since g, γ is not a bad pair and $R^-(B) \subseteq W$.

Since K is a 3^+-blocker it contains vertices $b \in B^+$, $\omega \in W^-$ with $r_K(b, \omega)$. Lemma 7.3 shows $r(b, \omega)$. Similarly there is a vertex $\gamma' \in G^- - \gamma$ with $r_K(g, \gamma')$. The uniqueness of the bad pair shows $\neg r(w, \gamma')$, so $r(g, \gamma')$. The relations $r(g, \gamma')$, $r(w, \gamma)$ and $r(b, \omega)$ show Corollary 7.2(ii) applies and we are done.

Now we can assume there is a bad pair with only one of its vertices in the augment. Suppose the sink is in the augment. Then there are vertices $w, w' \in W^+$, $\gamma \in G^-$, $\beta \in B^-$ such that

$$r(w', \gamma) \text{ and } r(w, \beta),$$

where the augment is for γ, w. The assumed bad pair is w', γ so $w' \in W$. This implies $\beta \in B$ since K is a 3^+-blocker. The assumptions of Cases 2 and 2.2 imply $w \in W$. Finally $\gamma \in G$ since $\gamma \in \overline{B}$ (for a bad pair) and $\gamma \in \overline{W}$ (by the augment).

Since K is a 3^+-blocker it contains vertices $b \in B^+$, $\omega \in W^-$ with $r_K(b, \omega)$. Lemma 7.3 shows $r(b, \omega)$, since $\neg r_K(w', \omega)$. Similarly there are vertices $g \in G^+$, $\gamma' \in G^- - \gamma$ with $r(g, \gamma')$. The relations $r(g, \gamma')$, $r(w', \gamma)$ and $r(b, \omega)$ show Corollary 7.2(ii) applies and we are done.

The remaining case is when there is a bad pair having only its source in the augment. There are vertices $g \in G^+$, $\overline{\omega} \in \overline{W}^-$, $w \in W^+$, $\overline{\beta} \in \overline{B}^-$ such that

$$r(g, \overline{\omega}) \text{ and } r(w, \overline{\beta}),$$

where the augment is for $\overline{\omega}, w$. $w, \overline{\beta}$ is the assumed bad pair. $g \in G$ because $g \notin B$ (since $R^-(B) \subseteq W$) and $g \notin W$ (since K is a 3^+-blocker).

Since K is a 3^+-blocker there are vertices $w' \in W^+ - w$, $\beta \in B^- - \overline{\omega}$ with $r_K(w', \beta)$. Lemma 7.3 with $\neg r_K(w', \overline{\beta})$ shows

$$r(w', \beta).$$

The pair $\overline{\omega}, w'$ is compatible, so augment it to get the graph L. We will show L has the desired properties. We consider two cases depending on the color of $\overline{\beta}$.

Suppose every possible choice for $\overline{\beta}$ is colored W. Since K is a 3^+-blocker there is a vertex $\gamma \in G^- - \overline{\omega}$ with $r_K(g, \gamma)$. The assumption on $\overline{\beta}$ shows $r(g, \gamma)$. Graph L has $r_L(g, \gamma)$ and $r_L(w, \overline{\beta})$. Since $\overline{\beta} \in W$, Lemma 6.1(i) shows we are done.

Suppose we can choose $\overline{\beta} \in G$. The hypothesis of Case 2.2 shows there are vertices $b \in B^+$, $\omega \in W^-$ with $r(b, \omega)$. Graph L has $r_L(g, \beta)$, $r_L(b, \omega)$ and $r_L(w, \overline{\beta})$. Since $\overline{\beta} \in G$, Lemma 7.1(iii) gives the desired conclusion. $\square$

It remains to consider when an augment of H gives a graph K that is a proper 3^--blocker. Lemma 4.2 shows the reverse of K is a proper 3^--blocker resulting from an augment. So the above lemma shows a different augment gives a graph L that is not a 2- or 3-blocker. The reverse of L is the result of the reverse augment, and is also not a 2- or 3-blocker.

4-blockers

Lemma 7.7 *If an augment of H gives a graph K that is a proper 4-blocker, a different augment of H gives a graph L that is a non-blocker or a 1-blocker.*

Proof: Suppose K is a proper 4-blocker for B. In H color B has $\Phi-1$ or $\Phi-2$ extreme vertices, since the augment decreases Φ by one. We consider these cases separately.

Case 1. B has $\Phi-1$ extreme vertices.

Lemma 7.4 shows we can assume $r(B)$. Hence the augment producing K involves a black extreme vertex. By possibly changing to the reverse graph assume it involves a black source. Thus there are vertices $w\in W^+$, $\overline{\beta}\in\overline{B}^-$, $b\in B^+$, $\beta\in B^-$ with

$$r(w,\overline{\beta}) \text{ and } r(b,\beta).$$

The relation $r_K(w,\beta)$ implies $w\in\overline{B}$, so without loss of generality take $w\in W$, making K a 4-blocker for B,W.

Lemma 4.4 shows there are vertices $g\in G^+, \gamma\in G^- - \overline{\beta}$ with $r_K(g,\gamma)$. Lemma 7.3 shows $r(g,\gamma)$, since $\neg r_K(g,\beta)$. Augment the compatible pair γ,w. In the resulting graph L, $r_L(b,\beta)$, $r_L(g,\overline{\beta})$ and B has Φ extreme vertices. Taking the colors as B and $\overline{B}$, L is not a blocker.

Case 2. B has $\Phi-2$ extreme vertices.

In this case the augment involves two non-black extremes. Note this implies $\neg r(B)$. As before assume K is a 4-blocker for B,W.

Case 2.1. $R^-(B)\cup R^+(B)\not\subseteq W$.

Together with the preceding observation this implies $r(B,G)$ or $r(G,B)$. By possibly changing to the reverse graph assume $r(B,G)$. Hence there are vertices $b\in B^+$, $\gamma\in G^-$, $w\in W^+$, $\omega\in W^-$ with

$$r(b,\gamma) \text{ and } r(w,\omega).$$

Here vertices b,γ come from the assumption $r(B,G)$, and the augment is for γ,w. $\omega\in W$ since K is a 4-blocker.

Next we show there are vertices $w'\in W^+-w$, $\beta\in B^-$, $g\in G^+$, $\gamma'\in G^- -\gamma$ with

$$r(w',\beta) \text{ and } r(g,\gamma').$$

Vertices satisfying these relations in graph K exist since K is a proper 4-blocker. Lemma 7.3 with $\neg r_K(b,\{\beta,\gamma'\})$ shows the same relations hold in H.

Augment the compatible pair β,w. The resulting graph L has $r_L(g,\gamma')$ and $r_L(w',\omega)$. Lemma 6.1(i) shows L is either a non-blocker, a 1-blocker, or a 4-blocker for B. The last alternative does not hold since B has $\Phi-2$ extreme vertices in L.

Case 2.2. $R^-(B)\cup R^+(B)\subseteq W$.

There are vertices $b\in B^+$, $\omega\in W^-$, $w\in W^+$, $\beta\in B^-$, $g\in G^+$, $\gamma\in G^-$ with

$$r(b,\omega),\ r(w,\beta) \text{ and } r(g,\gamma).$$

The first two relations and their four vertices come from Case 2.2. The two gray vertices exist and satisfy $r_K(g,\gamma)$, since K is a proper 4-blocker. We can assume $r(g,\gamma)$, since the augment involves a gray source or sink.

Suppose there are vertices $g' \in G^+ - g$, $\gamma' \in G^- - \gamma$ with $r(g',\gamma')$. Corollary 7.2(iii) shows we are done.

Next suppose there are vertices $g' \in G^+ - g$, $\omega' \in W^-$ with $r(g',\omega')$. This relation plus $r(w,\beta)$ and $r(g,\gamma)$ satisfy the hypothesis of Corollary 7.2(ii), so we are done. We can apply the same argument to the reverse graph if there are vertices $w' \in W^+$, $\gamma' \in G^- - \gamma$ with $r(w',\gamma')$.

Note that $|G^+| + |G^-| \geq 3$ since $|B^+| + |B^-| + |W^+| + |W^-| \leq (\Phi - 2) + (\Phi - 1) = 2\Phi - 3$. By possibly changing to the reverse graph assume $|G^-| \geq 2$. Every vertex of $G^- - \gamma$ has g as its only source predecessor, since otherwise one of the previous cases applies. Now the same reasoning extends this remark to every vertex of G^- (including γ). If $|G^+| \geq 2$ then one of the previous cases applies (specifically the case for $r(g',\omega')$). Thus $|G^+| = 1$. Since H is not a 3^+-blocker we conclude $r(W)$.

The preceding paragraph shows there are vertices $w' \in W^+$, $\omega' \in W^-$ and $\gamma' \in G^- - \gamma$ with

$$r(w',\omega') \text{ and } r(g,\gamma').$$

(Possibly $\omega' = \omega$.) Augment the compatible pair γ', b. For the resulting graph L, $r_L(g,\gamma)$, $r_L(w',\omega')$, and B has $\Phi - 2$ extreme vertices. Lemma 6.1(i) shows we are done. □

Theorem 7.8 *Any dag has $OPT = \Phi$ unless it is a blocker, in which case $OPT = \Phi + 1$.*

Proof: If H is a blocker then $OPT = \Phi + 1$ by Lemmas 4.3 and 2.9. Suppose H is not a blocker. $OPT = \Phi$ unless $\Phi = |V^-| = |V^+| > |B^+| + |B^-|$ for every $B \in \mathcal{P}$, by Lemmas 4.5 and 3.5. For the remaining case, Lemma 6.2 reduces an arbitrarily colored H to a tricolored dag. Lemmas 7.4 – 7.7 prove the theorem on tricolored dags in the remaining case. If the given dag has an isolated vertex the theorem still applies, as in the proof of Lemma 5.3. □

8 Algorithms

This section gives algorithms to solve the bipartite strong-connectivity augmentation problem in time $O(n+m)$ and the general augmentation problem in time $O((n+m)\log n)$. Here n and m denote the number of vertices and edges in the input graph, respectively. The main ingredient is an algorithm to find compatible pairs. This is given in Section 8.1. The rest of the algorithm consists of procedures that follow the proofs already presented. These procedures are given in Sections 8.2 – 8.5.

We start with two general observations. First recall that the strong component graph of a given digraph can be constructed in linear time, using routines for depth-first search and contraction [5]. Next observe that $OPT \leq \Phi + 1 = O(n)$ new edges get added to the graph. Hence all graphs we work with contain $O(n+m)$ edges.

8.1 Compatible pairs

This section describes several routines to find compatible pairs. A *compatible pair sequence* x_i^-, y_i^+, $i = 1, \ldots, k$ in a dag H has $x_i^- \in V^-$, $y_i^+ \in V^+$ with x_i^-, y_i^+ a compatible pair in the strong component dag of $H+\{(x_j^-, y_j^+) : 1 \le j < i\}$. Consider two sets $X^- \subseteq V^-$, $Y^+ \subseteq V^+$ that have equal cardinality and are consistently-colored. We give an algorithm to find a compatible pair sequence whose vertices are in $X^- \cup Y^+$ and whose length is $|X^-| - 1 = |Y^+| - 1$.

The algorithm is based on a procedure ONE-PAIR that finds the next pair in the desired sequence. Suppose we have found compatible pairs (x_j^-, y_j^+), $j = 1, \dots, i-1$. Let H_i be the strong component graph of $H + \{(x_j^-, y_j^+) : 1 \le j < i\}$. Throughout this discussion let V^-, V^+, X^- and Y^+ refer to H_i. Thus V^- and V^+ represent the sinks and sources of H_i. X^- and Y^+ represent the vertices of these sets V^- and V^+ that are in the original sets X^- and Y^+. The algorithm maintains a mapping $\sigma : X^- \cup Y^+ \to V^- \cup V^+$ with the property that for each source $z^+ \in Y^+$ and each sink $z^- \in X^-$,

$$r(z^+, \sigma(z^+)) \text{ and } r(\sigma(z^-), z^-).$$

The following procedure finds a compatible pair x^-, y^+ with $x^- \in X^-$, $y^+ \in Y^+$ and augments it. The procedure assumes $|X^-| = |Y^+| \ge 2$. It uses the above mapping σ and updates σ to make it valid in the augmented graph.

Procedure ONE-PAIR

Step 1. Choose distinct vertices $v_i^- \in X^-$, $i = 1, 2$ having $|\sigma^{-1}(v_i^-)| \le 2$. Similarly choose distinct vertices $w_i^+ \in Y^+$, $i = 1, 2$ having $|\sigma^{-1}(w_i^+)| \le 2$.

Step 2. This step chooses x^- as some v_i and y^+ as some w_j so that after possibly modifying σ,

$$\sigma(x^-) \ne y^+ \text{ and } \sigma(y^+) \ne x^-. \tag{3}$$

If for some $i \in \{1, 2\}$, $\sigma(v_i^-) \ne w_i^+$ and $\sigma(w_i^+) \ne v_i^-$, take (x^-, y^+) to be (v_i^-, w_i^+). Otherwise redefine $\sigma(v_1^-) = w_1^+$, $\sigma(w_2^+) = v_2^-$ and take (x^-, y^+) to be (v_1^-, w_2^+).

Step 3. Augment the compatible pair x^-, y^+. Update σ for the new graph H_i as follows: For every $z^+ \in \sigma^{-1}(x^-) - y^+$, change $\sigma(z^+)$ to $\sigma(y^+)$. Similarly for every $z^- \in \sigma^{-1}(y^+) - x^-$, change $\sigma(z^-)$ to $\sigma(x^-)$.

We now show that ONE-PAIR is correct. First note that the vertices of Step 1 exist. In proof take v_i^-, $i = 1, 2$ to be the two vertices of X^- with smallest value $|\sigma^{-1}(v_i^-)|$. Since the average value of $|\sigma^{-1}(v_i^-)|$, $v_i \in X^-$ is at most $|Y^+|/|X^-| = 1$, we have $|\sigma^{-1}(v_1^-)| + |\sigma^{-1}(v_2^-)| \le 2$. Hence these vertices have the desired property.

It is clear that Step 2 chooses (x^-, y^+) so (3) holds. Also note that if Step 2 redefines σ we have $r(w_i^+, v_i^-)$ for $i = 1, 2$, so the defining property of σ is maintained.

(3) implies x^-, y^+ is compatible by the defining property of σ. Hence the augment in Step 3 is valid. Finally we show Step 3 updates σ correctly. Recall from Lemma 2.6 that the new graph H_i has sources $V^+ - y^+$ and sinks $V^- - x^-$. Hence Step 3 updates all vertices whose σ value is no longer an extreme vertex. The update is correct by (3).

We conclude that ONE-PAIR is correct. This procedure can be repeated to find a compatible pair until $|X^-| = |Y^+| = 1$.

Now we show how to find the desired compatible pair sequence in linear time $O(m + n)$. We use these data structures: The values $\sigma(z)$ are stored in an array. Each vertex $z \in X^- \cup Y^+$ has a list of the vertices in $\sigma^{-1}(z)$. There is a list $SHORT[X^-]$ containing all vertices $x^- \in X^-$ that have $|\sigma^{-1}(x^-)| \le 2$, and a similar list $SHORT[Y^+]$. All lists are doubly-linked.

The mapping σ can be constructed initially in time $O(m + n)$ by two depth-first searches: The first labels each vertex of Y^+ with a sink that it can reach. The second labels each vertex of X^- with a source that can reach it. Finally we initialize the $SHORT$ lists.

Now we implement ONE-PAIR in time $O(1)$. (This implies the desired time bound for finding the compatible pair sequence.) Step 1 chooses vertices from the appropriate $SHORT$ list. In Step 2, suppose $\sigma(x^-) = z^+ \in Y^+$. Delete x^- from $\sigma^{-1}(z^+)$; if this makes $|\sigma^{-1}(z^+)| \leq 2$ add z^+ to $SHORT[Y^+]$. Process y^+ similarly. Step 3 changes the appropriate σ-values and adds vertices to the lists $\sigma^{-1}(z)$ for $z = \sigma(x^-), \sigma(y^+)$. This may cause these vertices to be deleted from the $SHORT$ lists. Note that Steps 1 and 2 imply $|\sigma^{-1}(x^-)|,\ |\sigma^{-1}(y^+)| \leq 2$. Hence Step 3 changes $O(1)$ σ values.

We extend this algorithm using the following observations.

Lemma 8.1 *Take any vertex $b \in B^+$ and let C be the set of all vertices compatible with b. Let $S = R^-(b)$ and $T = R^-(V^+ - b) - B$. If $|S| > 1$ then $C = T$. If $|S| = 1$ then $C = T - S$.* □

Lemma 8.2 *A pair of vertices x^-, y^+ that is compatible after an augment was compatible before the augment.*

Proof: Suppose x^-, y^+ was incompatible before the augment, i.e., before the augment we have $R^-(y^+) = \{x^-\}$ or $R^+(x^-) = \{y^+\}$. Suppose $R^-(y^+) = \{x^-\}$. The edge added by the augment is directed from a sink other than x^-, since x^- is a sink after the augment. Thus $R^-(y^+) = \{x^-\}$ after the augment. Similarly the condition $R^+(x^-) = \{y^+\}$ is preserved by the augment. □

In the given dag H let U be an arbitrary set of vertices. A *maximal compatible pair sequence for* U is a compatible pair sequence with all its vertices in U that cannot be extended by another pair. We will give an algorithm to find such a sequence. We start with an algorithm C-SEQUENCE that finds a maximal compatible pair sequence for U when the sources and sinks of U are consistently-colored.

Procedure C-SEQUENCE(U)

Step 1. Define sets $X^- \subseteq U^-$ and $Y^+ \subseteq U^+$ arbitrarily except for the constraint that each has cardinality $\min\{|U^-|,\ |U^+|\}$. If this cardinality is 0 return. Otherwise apply the algorithm given above to find and augment a compatible pair sequence with $|X^-| - 1$ pairs. (In the resulting graph at least one of the sets U^-, U^+ has cardinality one. Hence there may be one more compatible pair that can be augmented, which is found in Step 2.)

Step 2. Let u be the unique vertex of U^+ (U^-). Apply Lemma 8.1 to find the vertices of U compatible with u. If such a vertex exists augment the corresponding pair.

Next we show how to find a maximal compatible pair sequence for U when the extreme vertices of U have two colors, say B and W. First execute C-SEQUENCE($(U^- \cap B) \cup (U^+ \cap W)$). Then execute C-SEQUENCE($(U^- \cap W) \cup (U^+ \cap B)$). This procedure is correct by Lemma 8.2.

Finally we give an algorithm MAX-SEQUENCE(U) to find a maximal compatible pair sequence for U when the extreme vertices of U have three colors, B, W and G. Apply the above two-color procedure, taking the two colors as B and $\overline{B}$. Then apply the two-color procedure taking the two colors as W and $\overline{W}$.

Lemma 8.3 MAX-SEQUENCE(U) *finds a maximal compatible pair sequence for U in a tricolored dag in time $O(n + m)$.*

Proof: MAX-SEQUENCE is correct by Lemma 8.2 and the fact that any augment involves either a black vertex or a white vertex. For the time bound note that Step 2 of C-SEQUENCE finds the sets of Lemma 8.1 by depth-first search. As already noted the graph contains $O(n+m)$ edges. $\square$

We conclude this section by implementing the procedures for Section 2. The implementations for Proposition 2.1, Lemmas 2.2 and 2.3 are straightforward. To implement Lemmas 2.7 and 2.9 use MAX-SEQUENCE. In Lemma 2.9 if the graph is not bicolored, after the augments are executed we determine the first time a color B has $\Phi = |B^+| + |B^-|$. We undo all subsequent augments.

8.2 Bicolored dags

Lemmas 3.2 and 3.3.

To implement both these lemmas we must augment a maximal sequence of witness-compatible pairs, where the witnesses are $b^+ \in B^+$, $b^- \in B^-$ in Lemma 3.2 and $b^+ \in B^+$, $b^- \in B^-$, $w^+ \in W^+$, $w^- \in W^-$ in Lemma 3.3. We give a linear-time procedure for this task.

Define the initial two or four witnesses as in Lemma 3.2 or 3.3. Let X be the set of these initial witnesses. Execute MAX-SEQUENCE$(V - X)$. Then do the following for the two or four vertices of X. We illustrate for $b^- \in X$, the other witnesses are similar. Suppose in the current graph $V - X$ contains both a white source y^+ and a black sink y^-. These vertices are incompatible (by Lemma 8.2) so $r(y^+, y^-)$. Hence b^-, y^+ is witness-compatible. Augment this pair and (as in Lemmas 3.2 and 3.3) change b^- to y^-.

This procedure augments a maximal sequence of witness-compatible pairs. In proof let K be the dag at the end of the procedure, and let X' be the set of witnesses at the end. Lemma 8.2 shows that in K, $V - X'$ has no pairs that are compatible in the ordinary sense.

We claim that in K, $V - X'$ does not contain both a white source and a black sink. This is obvious if no augment was done for b^-, so suppose an augment was done. After executing MAX-SEQUENCE$(V - X)$ there is either a unique white source in $W^+ - X$ or a unique black sink in $B^- - X$ (by Lemma 2.5). The augment destroys that source or sink.

The claim implies that K does not have a witness-compatible pair b^-, y^+ for $y^+ \notin X'$. The only other possibility is that b^-, w^+ is witness-compatible. If it were there would be a white source $y^+ \notin X'$ and a black sink $y^- \notin X'$ with $r(y^+, b^-)$ and $r(w^+, y^-)$. But this makes y^-, y^+ compatible, a contradiction.

We conclude our procedure augments a maximal sequence of witness-compatible pairs. It is clear that the time is $O(m + n)$.

Lemma 3.5.

The algorithm starts by augmenting a maximal compatible pair sequence until the graph becomes bicolored or some color becomes big. This is done by executing MAX-SEQUENCE(V) and undoing the augments that occur after the first time the termination condition is achieved. If there are two big colors we reduce to a bicolored dag as shown in the lemma. If there is a unique big color B, augment a maximal compatible sequence where each augment involves a black vertex, by executing MAX-SEQUENCE(V) with the colors taken to be B and $\overline{B}$. If some augment creates a second big color, undo any subsequent augments and proceed as above.

8.3 Blockers

Detecting a blocker.

We decide if an arbitrarily-colored dag is a blocker in linear time as follows. A set $R^-(X)$ or $R^+(X)$ can be computed in linear time by depth-first search. Hence we can check if a dag is a blocker for a given color B in linear time. (For 3- and 4-blockers, color B determines W.) At most two colors can have Φ extreme vertices, so we can check for a 1-blocker in linear time. At most three colors of a proper 4-blocker can have $\Phi - 1$ extreme vertices (Lemma 4.4 implies $\Phi \geq 3$). So we can check for such a blocker in linear time. V^+ contains vertices of exactly three colors in a 3^+-blocker. Hence we can check for a 3-blocker in linear time. For a 2-blocker, find two vertices $x \in V^+$, $y \in V^-$ with $r(x,y)$. If the dag is a 2-blocker it must be a 2-blocker for the color of x or the color of y. So we can check for a 2-blocker in linear time.

Implementing Lemma 4.5 is trivial.

8.4 Core-bicolored dags

Lemma 5.2.

Call a pair $(x^+, x^-) \in \mathcal{B}$ *good* if $|B^+| + |B^-| \leq |V^+| - 2$ or $|B^+| + |B^-| = |V^+| - 1$ and at most one vertex of the pair belongs to G. A pair of $\mathcal{W}$ is good if it satisfies the symmetric condition. The proof of Lemma 5.2 shows if H is not a blocker then it has good pairs $(x^+, x^-) \in \mathcal{B}$, $(y^+, y^-) \in \mathcal{W}$ with all four vertices distinct. We give an algorithm to find two such pairs. Using these pairs we recolor vertices to reduce to the bicolored case, as shown in the lemma.

First note that we can find a good pair in $\mathcal{B}$ by depth-first search: If $|B^+| + |B^-| \leq |V^+| - 2$ we do a depth-first search from $(B \cup G)^+$. If $|B^+| + |B^-| = |V^+| - 1$ we do a depth-first search from B^+ and a second depth-first search from G^+.

Start by finding a good pair $(x^+, x^-) \in \mathcal{B}$. Then try to find a good pair of $\mathcal{W}$ in the graph $H - \{x^+, x^-\}$. If such a pair exists we are done. Otherwise execute the case below that applies.

Case 1. $H - x^+$ does not contain a good pair of $\mathcal{W}$.

This implies x^+ is in every good pair of $\mathcal{W}$. Let (x^+, y_1^-) be a good pair of $\mathcal{W}$. If it is the only good pair of $\mathcal{W}$ then use it with a good pair of $\mathcal{B}$ in $H - \{x^+, y^-\}$ as the desired pairs. Otherwise let (x^+, y_2^-) be a second good pair of $\mathcal{W}$. There is a good pair of $\mathcal{B}$ in $H - \{x^+\}$, say (z^+, z^-). Use (z^+, z^-) with (x^+, y_i^-), where $i \in \{1,2\}$ is chosen so $y_i^- \neq z^-$, as the desired pairs.

Case 2. Both $H - x^+$ and $H - x^-$ contain a good pair of $\mathcal{W}$, but $H - \{x^+, x^-\}$ does not.

This implies we can find good pairs (x^+, y^-) and (y^+, x^-) of $\mathcal{W}$, where $y^- \neq x^-$ and $y^+ \neq x^+$. $x^+, x^- \in G$ since $(x^+, x^-) \in \mathcal{B}$. If $y^- \in G$ then (x^+, y^-) is good for $\mathcal{B}$, so take (x^+, y^-) and (y^+, x^-) as the desired pairs. Now without loss of generality assume that both y^+ and y^- are white. There is a good pair of $\mathcal{B}$ in either $H - x^+$ or $H - x^-$. Find such a pair and use it with either (x^+, y^-) or (y^+, x^-) to form the desired pairs.

Theorem 8.4 *The bipartite strong-connectivity augmentation problem and the cable-framework rigidity problem can be solved in linear time.* □

8.5 General algorithm

Merging colors.

Lemma 6.2 is straightforward to implement in time $O(n+m)$. The algorithm finds a permissible merge into three colors that gives a non-blocker, after trying at most five merges.

The implementation of Lemma 6.3 is similar to Case 1 of Lemma 5.2 in the previous section.

Augmentation algorithm.

We are given a tricolored dag H that is not a blocker. We implement the proof of Section 7 to solve the strong connectivity augmentation problem on H. Recall that the proof orders the blocker types as 2,3,4,1. The relation $\prec$ corresponds to precedence in this list, e.g., $2 \prec 3, 4, 1$. Similarly for $\preceq, \succ, \succeq$.

For $i = 2, 3, 4$ define an *i-witness set* to be a set of vertices that satisfies one of the conditions of Lemma 7.1 or Lemma 6.1(i) that preclude H from being a blocker of type $\preceq i$. For instance the six vertices of Lemma 7.1(iv) form a 3-witness set, as well as a 2-witness set. Vertices $b^+ \in B^+$, $b^- \in B^-$, $x^+ \in \overline{B}^+$, $x^- \in \overline{B}^-$ with $r(b^+, b^-)$ and $r(x^+, x^-)$ form a 2-witness set, since they satisfy the condition of Lemma 7.1(i). A 4-witness set differs from the others because it has an additional condition: There is a distinguished color B has $\Phi - 2$ extreme vertices. For instance the above 3-witness set is a 4-witness set in a dag where B has $\Phi - 2$ extremes.

Each of Lemmas 7.4 – 7.7 is easily implemented in linear time. Lemma 7.4 reduces the graph to a bicolored dag. Lemmas 7.5 – 7.7 each start with a dag that has an augment giving an i-blocker ($i = 2, 3, 4$) and execute a different augment that is guaranteed not to give a blocker of type $\preceq i$. This guarantee holds because the lemma (and its corresponding algorithm) produces an i-witness set. (Lemma 7.7 produces a 4-witness set in a graph where B has $\Phi - 2$ extreme vertices.)

These observations show we can execute one augment in linear time. This implies an algorithm for the augmentation problem that runs in time $O((n+m)\Phi)$.

Now we give an algorithm with the improved time bound $O((n+m)\log \Phi)$. The algorithm consists of four stages: an initial stage and a stage for i-blockers, $i = 2, 3, 4$. We refer to the stage for i-blockers as *stage i*. The algorithm starts in the initial stage. After that it executes between zero and three blocker stages, always progressing from a stage i to a higher-numbered stage.

Each stage has as input a tricolored dag that is not a blocker. For the initial stage this is the given dag H. For subsequent stages it is an augmented version of H. Stage i is also given an i-witness set.

Each stage begins by executing a sequence of augments that end with a dag K, such that the following properties hold.

(i) Each augment decreases Φ by 1.

(ii) K is not a blocker.

(iii) No augment involves a vertex in the given witness set. (This set is $\emptyset$ in the initial stage.)

(iv) In stage 4 each augment involves a vertex of color B (so B always has $\Phi - 2$ extreme vertices).

(v) K is either (a) rooted, (b) bicolored, (c) has $\Phi = |B^+| + |B^-|$ for some color B, (d) has $\Phi = O(1)$, or (e) an augment for K has been found whose execution gives a blocker.

In cases (v.a) – (v.d) complete the solution to the augmentation problem in time $O(n+m)$ using the procedure for Lemmas 2.3, 3.4, 4.5 or the above $O((n+m)\Phi)$ algorithm, respectively. (Note K may be strongly connected, in which case it satisfies (v.b).)

Suppose case (v.e) holds. If some color has no source or no sink, execute the procedure of Lemma 6.3 to get a bicolored dag, and complete the solution as in the previous paragraph. Otherwise execute the procedure of Lemmas 7.4 – 7.7 to get a dag L that is not a blocker. (For example suppose K has an augment giving a 2-blocker. Lemma 7.5 may give an augment that produces a non-blocker, or perhaps a 4-blocker. In the first case we take the non-blocker as L. In the second case we execute the procedure for Lemma 7.7. Eventually we get a non-blocker L.) If L satisfies any of conditions (v.a) – (v.c) complete the solution as in the previous paragraph. Otherwise the last lemma whose procedure is executed gives a j-witness set for L, where $j \in \{2, 3, 4\}$. Proceed to stage j. (Note that if Lemma 7.4 for 1-blockers is executed, it gives a graph satisfying (v.b) so the solution gets completed.)

This algorithm executes each of the four stages at most once. In proof suppose stage i ends with (v.e) satisfied. The augment of (v.e) gives a blocker of type $\succ i$. This follows since the i-witness set is preserved (by (iii) – (iv)). Hence the procedure of Lemmas 7.4 – 7.7 gives a non-blocker L that, if not bicolored, has a j-witness set for some $j \succ i$. In this case the algorithm advances to stage j, so no stage gets repeated.

Now we show how to implement a stage. We need only describe the procedure that executes the sequence of augments satisfying (i) – (v).

Let X be the given set of i-witnesses ($X = \emptyset$ for the initial stage). The initial stage and stages 2 and 3 begin by executing MAX-SEQUENCE($V - X$). Stage 4 executes MAX-SEQUENCE($V - X$) in the dag with colors taken to be B and $\overline{B}$ (for the distinguished color B).

Let S be the sequence of augments that get executed in MAX-SEQUENCE. Undo these augments. Take the final dag K to be the result of executing augments of S, in order, until one of conditions (v.a) – (v.d) holds, or executing the next augment in S would give a blocker (so (v.e) holds).

Correctness of this procedure depends on two facts. First note that the final sequence of augments satisfies (i). This holds since an augment that does not decrease Φ is executed in a dag that has a color B with Φ extreme vertices, so (v.c) holds.

The second fact is that (v) is satisfied when the entire sequence S gets executed to produce K. We show that in this case (v.d) holds,$\Phi = O(1)$. Note that any i-witness set has $O(1)$ vertices (specifically it has at most 6 vertices). Hence the following lemma is sufficient.

Lemma 8.5 *Let H be a tricolored graph with $X \subseteq V$.*

(i) If MAX-SEQUENCE($V - X$) *results in a dag K with $\Phi = |V^+| = |V^-|$ then K has $\Phi \le |X| + 3$.*

(ii) Suppose executing MAX-SEQUENCE($V - X$) *in the dag with colors taken to be B and $\overline{B}$ results in a dag K that (using the original colors) has $\Phi = |V^+| = |V^-| = |B^+| + |B^-| + 2$. Then K has $\Phi \le |X| + 4$.*

Proof: (i) Consider K. If every color B has $|(B - X)^+| \le 1$ then $\Phi = |V^+| \le |X| + |V^+ - X| \le |X| + 3$. Hence assume some color B has $|(B - X)^+| > 1$. This implies $|(\overline{B} - X)^-| \le 1$ (Lemma 2.5). If $|(B - X)^-| \le 1$ then $\Phi = |V^-| \le |X| + |V^- - X| \le |X| + 2$. So assume $|(B - X)^-| > 1$. This implies $|(\overline{B} - X)^+| \le 1$ (Lemma 2.5). Hence $\overline{B}$ contains at most $|X| + 2$ extreme vertices. But $\overline{B}$ contains at least Φ extreme vertices (since B contains at most Φ extremes). We have deduced $\Phi \le |X| + 2$.

(ii) If $|(B - X)^+|, |(B - X)^-| \le 1$ then $\Phi = |B^+| + |B^-| + 2 \le |X| + 4$. So without loss of generality assume $|(B - X)^+| > 1$. The rest of the argument proceeds as in part (i). □

The efficiency of a stage depends on how fast we can find the subsequence of augments of S that get executed. We start by finding the first augment in S that results in a dag satisfying any of the conditions (v.a) – (v.c). This is easy to do in linear time. Let S' be the sequence S up to and including the augment just found. Now we must find the first augment in S' that produces a blocker. This can be done by a binary search of S', where each probe tests if the corresponding dag is a blocker.

To see this binary search is correct note that the first augment in S' that produces a blocker changes a dag with $OPT = \Phi$ into a dag with $OPT = \Phi+1$. Every subsequent dag has $OPT = \Phi+1$ (since each augment decreases Φ by 1, and an augment decreases OPT by at most 1). Hence every subsequent dag is a blocker.

Each probe of the binary search takes time $O(n+m)$ (Section 8.3). Thus the time for the binary search is $O((n+m)\log\Phi)$.

Theorem 8.6 *The problem of strong-connectivity augmentation with partition constraints can be solved in time $O((n+m)\log n)$ (the value of OPT can be found in time $O(n+m)$).* $\square$

9 Related problems

The weighted version of bipartite strong-connectivity augmentation is NP-hard, by reduction from the bipartite Hamiltonian cycle problem.

[3] solves the problem of making an undirected bipartite graph k-edge-connected while preserving bipartiteness, i.e., bipartite k-edge-connectivity augmentation. This paper has solved the $k = 1$ case of bipartite directed k-edge-connectivity augmentation. We believe this directed problem is harder than the undirected problem of [3]. As evidence first recall that the directed problem for $k = 1$ already has a richer set of blockers than the undirected problem for arbitrary k. Also for $k = 1$ the directed problem can have $OPT > \Phi$ for graphs with arbitrarily large values of OPT. In contrast [3] shows that for the undirected problem, $OPT \geq 2k + 1$ implies $OPT = \Phi$. Finally it can be shown that in the directed problem with arbitrary k, OPT takes on values between Φ and $\Phi + k$ (for an appropriate definition of Φ). In contrast [3] shows any undirected bipartite graph has OPT equal to Φ or $\Phi + 1$.

We close by noting that if our operation is not adding new cables but replacing existing cables by rods, the rigidity problem can be solved by the Lucchesi-Younger Theorem [7].

References

[1] J.A. Baglivo and J.E. Graver, Incidence and Symmetry in Design and Architecture, Cambridge Univ. Press, Cambridge, 1983.

[2] J. Bang-Jensen and T. Jordán, *Edge-connectivity augmentation preserving simplicity*, Proc. 38th Annual Symp. on Foundations of Computer Science(1997), 486-495; also SIAM J. Disc. Math., to appear.

[3] J. Bang-Jensen, H. N. Gabow, T. Jordán and Z. Szigeti, *Edge-connectivity augmentation with partition constraints*, Proc. 9th Annual ACM-SIAM Symp. on Disc. Algorithms, 306-315, 1998 (also available as Preprint No. 32, Department of Mathematics and Computer Science, Odense University, 1997, and at `http://www.cs.colorado.edu/~hal`); and SIAM J. Disc. Math., to appear.

[4] E.D. Bolker and H. Crapo, How to brace a one-story building, Environ. Plan. B, 4, 125-152, 1977.

[5] T.H. Cormen, C.E. Leiserson and R.L. Rivest, Introduction to Algorithms, McGraw-Hill, New York, 1990.

[6] K.P. Eswaran and R. E. Tarjan, Augmentation problems, SIAM J. Computing 5,4, 653-665, 1976.

[7] A. Frank, Connectivity augmentation problems in network design, in: Mathematical Programming: State of the Art 1994, (Eds: J.R. Birge and K.G. Murty), The University of Michigan, Ann Arbor, MI, 34-63, 1994.

[8] M. Grötschel, C.L. Monma and M. Stoer, Design of survivable networks, in: Handbook in Operations Research and Management Science Vol. 7, Network Models (M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser, eds.) North-Holland, Amsterdam, 617-672, 1994.

[9] T-S. Hsu and M-Y. Kao, Optimal augmentation for bipartite componentwise biconnectivity in linear time, Algorithms and Computation (Proc. ISAAC '96), Springer Lecture Notes in Computer Science 1178, 213-222, 1996.

[10] T. Ishii, H. Nagamochi and T. Ibaraki, Optimal augmentation to make a graph k-edge-connected and triconnected, Proc. 9th Annual ACM-SIAM Symp. on Disc. Algorithms, 280-289, 1998.

[11] T. Jordán, Two NP-complete augmentation problems, Preprint no. 8, Department of Mathematics and Computer Science, Odense University (1997).

[12] G. Kant and H.L. Bodlaender, Planar graph augmentation problems, Algorithms and Data Structures (Proc. WADS '91), Springer Lecture Notes in Computer Science 519, 286-298, 1991.

[13] H. Nagamochi and P. Eades, *Edge-splitting and edge-connectivity augmentation in planar graphs*, Integer Programming and Combinatorial Optimization (Proc. 6th Conf. on Int. Prog. and Comb. Opt.), Springer Lecture Notes in Computer Science 1412, pp. 96–111, 1998.

[14] A. Recski, Matroid Theory and its Applications in Electric Network Theory and in Statics, Akadémiai Kiadó, Budapest, 1989.

[15] M. Stoer, Design of Survivable Networks, Ph.D. Thesis, Springer Lecture Notes in Mathematics 1531, 1992.