Incrementing Bipartite Digraph Edge-connectivity

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Abstract

This paper solves the problem of increasing the edge-connectivity of a bipartite digraph by adding the smallest number of new edges that preserve bipartiteness. A natural application arises when we wish to reinforce a 2-dimensional square grid framework with cables. We actually solve the more general problem of covering a crossing family of sets with the smallest number of directed edges, where each new edge must join the blocks of a given bipartition of the elements. The smallest number of new edges is given by a min-max formula that has six infinite families of exceptional cases. We discuss a problem on network flows whose solution has a similar formula with three infinite families of exceptional cases. We also discuss a problem on arborescences whose solution has five infinite families of exceptions. We give an algorithm that increases the edge-connectivity of a bipartite digraph in the same time as the best-known algorithm for the problem without the bipartite constraint: $O(km \log n)$ for unweighted digraphs and $O(nm \log(n^2/m))$ for weighted digraphs, where n, m and k are the number of vertices and edges of the given graph and the target connectivity, respectively.

1 Introduction

In the *bipartite digraph edge-connectivity incrementation problem* we are given a digraph whose undirected version is bipartite. We wish to increase the edge-connectivity by adding the smallest number of new edges that keep the graph bipartite. Fig. 1 shows an example. Let k be any nonnegative integer. The graph of Fig. 1(a) is k-edge-connected. Adding three edges gives the $(k+1)$-edge-connected graph of Fig. 1(b). It is easy to see that adding only two edges does not achieve $(k+1)$-edge-connectivity unless the graph is made nonbipartite (in which case two edges suffice).

A motivation for this problem comes from "square-grid frameworks" in statics. (We will not attempt a complete discussion of these ideas here, see [12].) These frameworks can be braced by diagonal "rods" or cheaper "cables." Baglivo and Graver [2] show that a framework with cables (and perhaps rods) has a natural representation as a bipartite digraph, such that the framework is "rigid" if and only if its bipartite digraph is strongly connected. Thus a framework that is rigid and remains so even after any k or fewer cables fail is precisely a framework whose digraph is $(k+1)$-edge-connected. The *cable-framework reinforcement problem* is to add the smallest number of new cables to a given framework so it tolerates an extra cable failure, i.e., if the original framework remains rigid after any k cables fail, the new framework remains rigid after any $k+1$ failures. Clearly this problem is equivalent to bipartite digraph edge-connectivity incrementation. (We note that the weighted version of either of these problems is NP-hard, by reduction from the bipartite Hamiltonian cycle problem.)

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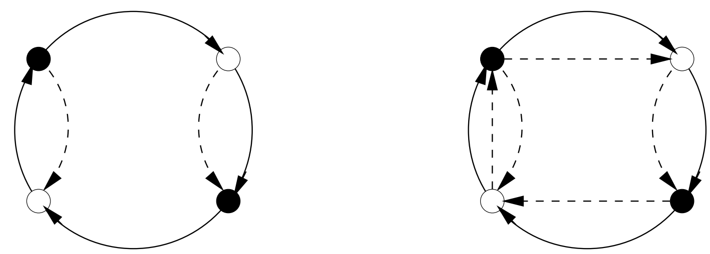


(a) (b)

Figure 1: Bipartite digraph (a) and solution to its edge-connectivity incrementation problem (b). A directed line represents an edge that has multiplicity k if it is drawn solid and multiplicity 1 if it is drawn dashed.

We solve a generalization of the (unweighted) digraph problem. Consider a crossing family $\mathcal{I}$ on a set of elements V, along with a partition $\mathcal{P}$ of V into two blocks. A directed edge xy is said to *cover* any set $X \in \mathcal{I}$ that contains y but not x. In the *bipartitioned crossing-family covering problem* we wish to cover every set of $\mathcal{I}$ using the fewest number of directed edges, each of which joins the two blocks of $\mathcal{P}$.

If a digraph has edge-connectivity k, the vertex sets of in-degree k form a crossing family. Hence the crossing-family problem generalizes the edge-connectivity incrementation problem.[1] In particular we can start with a general (not necessarily bipartite) digraph, and a given partition of the vertices into two sets. The goal is to add the fewest number of new directed edges that respect the bipartition and increase the edge-connectivity by one.

Our main result shows that the optimum number of edges OPT has a characterization similar to ones given in [3] and [8] for related connectivity problems. These characterizations say that $OPT = \Phi$ unless the graph is a "blocker," in which case $OPT = \Phi + 1$. Here Φ is a natural lower bound coming from a subpartition of the vertex set. A "blocker" is a graph belonging to one of a number of infinite families. [3] solves the problem of undirected bipartite graph edge-connectivity augmentation. This problem has one family of blockers. [8] solves the problem of making a bipartite digraph strongly connected. Here there are four families of blockers. Our connectivity result generalizes [8], which corresponds to increasing the edge-connectivity of a 0-edge-connected bipartite graph. Our theorem has six families of blockers, the first four being direct analogs of those of [8]. In fact we show the two new families of blockers never exist when the given crossing family is a ring family. For connectivity incrementation we have a ring family when the initial graph is 0-edge connected but not for any higher connectivity.

Note that [3] solves the problem of connectivity augmentation, where any target connectivity can be given. In contrast our results only increase the connectivity by 1. [9] investigates the edge-connectivity augmentation problem for bipartite digraphs. An algorithm is given that finds a solution using at most k edges more than optimum. The directed augmentation problem appears to be more difficult than the undirected version solved in [3]. As evidence we can contrast our solution to the incrementation problem with [3]. As already mentioned the directed problem has more blockers. Also a blocker for undirected incrementation exists only when $\Phi = 2$, whereas in

[1] When $k > 0$ the bipartition $\mathcal{P}$ can be inferred from the given graph. When $k = 0$ this may not be possible, so we assume $\mathcal{P}$ is given in the edge-connectivity incrementation problem. This assumption holds for the application to cable-framework reinforcement.

the directed case blockers exist for any $\Phi \geq 2$. Section 6 gives more details of this comparison.

We present two other incrementation problems that have a similar structure. Specifically we present a problem on network flows that has the same $\Phi/\Phi+1$ characterization of OPT and has three families of blockers. We present a problem on spanning arborescences that has five families of blockers. Both of these problems are instances of bipartitioned crossing-family covering. We suspect there are similar blocker-type characterizations for other problems in related areas.

We also give a generalized version of our characterization of OPT. It includes Frank's min-max formula for digraph edge-connectivity augmentation [5] when specialized to connectivity incrementation.

Finally we present an algorithm for bipartite digraph edge-connectivity incrementation (allowing nonbipartite starting graphs). The algorithm has the same time bound as the best-known algorithm for incrementation without the bipartite constraint, $O(km \log n)$ for unweighted digraphs and $O(nm \log(n^2/m))$ for weighted digraphs, for k the target connectivity. Here n, m and k are the number of vertices and edges of the given graph and the target connectivity, respectively. "Weighted" and "unweighted" refer to the graph representation: In an unweighted digraph parallel edges are allowed but each copy is stored separately, and each copy contributes to m. In a weighted digraph each distinct edge has an integral weight that gives the number of parallel copies; each edge contributes 1 to m independent of the number of copies.

Section 2 discusses basic facts about crossing families and gives the main tools for our problem. Section 3 proves our incrementation theorem for certain crossing families. Section 4 defines blockers and establishes their properties. Section 5 proves our incrementation theorem in general by reducing it to the families of Section 3. Section 6 compares our theorem with [3] and [8]. It also gives our generalized incrementation theorem, and the flow and arborescence problems. Section 7 gives the algorithm. We close this section with some definitions.

Consider a finite universe V. We use $\subseteq$ for set containment and $\subset$ for proper set containment. For an element $v \in V$ we usually abbreviate the singleton set $\{v\}$ to v. For example $S \cup v$ denotes $S \cup \{v\}$ and for a function f defined on subsets of V, $f(v)$ denotes $f(\{v\})$.

Let X, $Y \subseteq V$. $\overline{X}$ denotes the complement set $V - X$. X and Y are *co-disjoint* if $\overline{X}$ and $\overline{Y}$ are disjoint, i.e., $X \cup Y = V$. X and Y are *intersecting* if $X \cap Y$, $X - Y$ and $Y - X$ are all nonempty sets. X and Y are *crossing* if they are intersecting and furthermore $V - (X \cup Y)$ is nonempty, i.e., they are not co-disjoint. Let $\mathcal{F}$ be a family of subsets of V. $\mathcal{F}$ is a *ring family* if it is closed under intersection and union. $\mathcal{F}$ is an *intersecting* (*crossing*) *family* if $X \cap Y$, $X \cup Y \in \mathcal{F}$ whenever X, $Y \in \mathcal{F}$ and X and Y are intersecting (crossing) sets.

In a digraph G for $S \subseteq V$, an edge xy with $x \in S$ and $y \notin S$ *leaves* S and *enters* $V - S$. $\rho(S)$ ($\delta(S)$) denotes the number of edges entering (leaving) S.

In the bipartitioned crossing-family covering problem we make the following assumptions to rule out trivial cases:

(a) $\mathcal{I} \neq \emptyset$;
(b) $\emptyset, V \notin \mathcal{I}$ and both blocks of $\mathcal{P}$ are nonempty.

If (a) fails then no new edges are needed. If (b) fails then no covering exists. We use OPT to denote the fewest number of edges in a cover. Unless stated otherwise when discussing the bipartitioned crossing-family covering problem we assume that an edge respects the bipartition (since an edge joining vertices in the same block of $\mathcal{P}$ is irrelevant).

2 Basic notions

For any family of sets $\mathcal{F}$, an *$\mathcal{F}$-set* is a set of $\mathcal{F}$. $\|\mathcal{F}\|$ denotes the maximum cardinality of a family of pairwise disjoint $\mathcal{F}$-sets. $\mu(\mathcal{F})$ denotes the set of all inclusionwise minimal $\mathcal{F}$-sets. Hence $\|\mu(F)\| = \|\mathcal{F}\|$. When $\mu(\mathcal{F})$ consists of exactly one set $\mu^*(\mathcal{F})$ denotes that set, i.e., $\mu(\mathcal{F}) = \{X\}$ exactly when $\mu^*(\mathcal{F}) = X$.

Let $\mathcal{O}$ denote the family of complements of $\mathcal{I}$-sets. $\mathcal{O}$ is a crossing family. An edge *covers* an $\mathcal{O}$-set O if it leaves O. So the given covering problem on $\mathcal{I}$ amounts to finding a minimum cardinality set of edges that cover $\mathcal{O}$, each edge joining two differently colored elements. This problem is equivalent to our covering problem if we start with the crossing family $\mathcal{O}$ as the given family. (A solution to one covering problem gives a solution to the other by reversing the edges.)

We will treat $\mathcal{I}$ and $\mathcal{O}$ symmetrically in this paper. For instance the definitions given below are stated in terms of the family $\mathcal{I}$. All the analogous definitions for $\mathcal{O}$ should be considered to hold. All of our lemmas and theorems have a *complement-symmetric* version obtained by interchanging every occurrence of $\mathcal{I}$ and $\mathcal{O}$. We do not explicitly state the complement-symmetric versions, and a citation of a lemma or theorem may actually refer to the complement-symmetric version. Henceforth $\mathcal{F}$ will refer to both families together, i.e., $\mathcal{F} = (\mathcal{I}, \mathcal{O})$.

2.1 Cores and zones

An *$\mathcal{I}$-core* is a set of $\mu(\mathcal{I})$.

Lemma 2.1 *For any crossing family $\mathcal{I}$, either the $\mathcal{I}$-cores are pairwise disjoint or any two $\mathcal{I}$-cores are intersecting and co-disjoint.*

Proof: Observe that any two $\mathcal{I}$-cores that are not disjoint are intersecting and co-disjoint (by minimality). Let $I_1, \ldots, I_r$ be a maximal collection of pairwise disjoint $\mathcal{I}$-cores. If the first alternative of the lemma fails there is another $\mathcal{I}$-core J that is not disjoint from some I_j, say I_1. So I_1 and J are co-disjoint. Since $J - I_1$ contains no $\mathcal{I}$-core, $r = 1$ and the second alternative of the lemma holds. □

The lemma shows $\|\mathcal{I}\|$ either equals 1 or $|\mu(\mathcal{I})|$. Our initial assumption $\mathcal{I} \neq \emptyset$ implies $\|\mathcal{I}\| \geq 1$ for the given $\mathcal{I}$.

Corollary 2.2 *If $\mathcal{I}$ has two intersecting cores then $\|\mathcal{I}\| = 1$ and $\|\mathcal{O}\| > 1$.*

Proof: Let I and J be intersecting $\mathcal{I}$-cores. The lemma implies $\|\mathcal{I}\| = 1$ and I and J are co-disjoint. Thus $\overline{I}$ and $\overline{J}$ are disjoint $\mathcal{O}$-sets, so $\|\mathcal{O}\| > 1$. □

Lemma 2.3 *An $\mathcal{I}$-set I intersecting with an $\mathcal{O}$-set O has $I - O$ an $\mathcal{I}$-set. Hence no $\mathcal{I}$-core is intersecting with an $\mathcal{O}$-set.*

Proof: Suppose an $\mathcal{I}$-set I is intersecting with an $\mathcal{O}$-set O. If I and O are co-disjoint then $I - O = \overline{O}$ is an $\mathcal{I}$-set. The other possibility is that I and O are crossing. Hence I and $\overline{O}$ are crossing $\mathcal{I}$-sets. Thus $I - O = I \cap \overline{O}$ is an $\mathcal{I}$-set. □

Lemma 2.4 *Two $\mathcal{O}$-sets with nonempty intersection are intersecting and co-disjoint if their union contains every $\mathcal{I}$-core, or if their intersection contains no $\mathcal{O}$-core.*

Proof: Let the $\mathcal{O}$-sets be N and O. First suppose $N \cup O$ contains every $\mathcal{I}$-core. Then $N \cup O$ is not an $\mathcal{O}$-set since its complement contains no $\mathcal{I}$-core. This implies $N \not\subseteq O$, $O \not\subseteq N$ and N and O are not crossing. We conclude that N and O are intersecting and co-disjoint. The argument when $N \cap O$ contains no $\mathcal{O}$-core is similar, using $N \cap O$ not an $\mathcal{O}$-set. □

We assume the partition $\mathcal{P}$ is specified by assigning one of two colors to each element of V. These two colors will be unnamed. However we use *black* as a variable name that ranges over the two given colors. Correspondingly B denotes the set of all black elements. A set of elements is called *black* if it is contained in B. When we need to refer to both colors of $\mathcal{P}$ we shall call them black and *white*, with W and *white sets* defined analogously. A set of elements is *mixed* if it contains at least one element of both colors.

We use several subfamilies of $\mathcal{I}$ and $\mathcal{O}$. For any color B, $\mathcal{I}_B$ is the family of black $\mathcal{I}$-sets. Hence the $\mathcal{I}_B$-cores are the black $\mathcal{I}$-cores. $\mathcal{I}_X$ is the family of $\mathcal{I}$-sets that do not contain a black or white $\mathcal{I}$-core. The $\mathcal{I}_X$-cores are the mixed $\mathcal{I}$-cores. Any $\mathcal{I}_X$-set is mixed, but a mixed $\mathcal{I}$-set need not be in $\mathcal{I}_X$. For any color B the family $\mathcal{I}_{B'}$ is $\mathcal{I}_W \cup \mathcal{I}_X$. Hence the $\mathcal{I}_{B'}$-cores are the nonblack $\mathcal{I}$-cores. We have noted that for $\mathcal{A} = \mathcal{I}_B, \mathcal{I}_X, \mathcal{I}_{B'}$, the $\mathcal{A}$-cores are the same as the $\mathcal{I}$-cores contained in $\mathcal{A}$.

Fix $\mathcal{F} = (\mathcal{I}, \mathcal{O})$ and let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of V. We now define the families

$$\mathcal{I}[\mathcal{A},\, \mathcal{B}] \text{ and } \mathcal{O}[\mathcal{A},\, \mathcal{B}].$$

We state the definition for $\mathcal{I}[\mathcal{A},\, \mathcal{B}]$, $\mathcal{O}[\mathcal{A},\, \mathcal{B}]$ is similar.

Usually we choose $\mathcal{A}$ and $\mathcal{B}$ from our families $\mathcal{I}_B$, $\mathcal{I}_X$, $\mathcal{I}_{B'}$ and the similar families with $\mathcal{I}$ replaced by $\mathcal{O}$ or B replaced by W. In all such cases $\mathcal{I}[\mathcal{A},\, \mathcal{B}]$ consists of all $\mathcal{I}$-sets that contain every $\mathcal{A}$-core and are disjoint from every $\mathcal{B}$-core. For example $\mathcal{I}[\mathcal{I}_B,\, \mathcal{O}_{B'}]$ is the family of all $\mathcal{I}$-sets that contain every black $\mathcal{I}$-core and are disjoint from every nonblack $\mathcal{O}$-core.

We also allow more general $\mathcal{A}$ and $\mathcal{B}$. For (possibly empty) families $\mathcal{I}_i \subseteq \mathcal{I}$, $\mathcal{O}_i \subseteq \mathcal{O}$, $i = 1, 2$, define $\mathcal{I}[\mathcal{I}_1 \cup \mathcal{O}_1,\, \mathcal{I}_2 \cup \mathcal{O}_2]$ as the family of all $\mathcal{I}$-sets that contain every $\mathcal{I}$-core in $\mathcal{I}_1$ and every $\mathcal{O}$-core in $\mathcal{O}_1$ and are disjoint from every $\mathcal{I}$-core in $\mathcal{I}_2$ and every $\mathcal{O}$-core in $\mathcal{O}_2$. This definition is consistent with the special case given above. In using this notation we will not explicitly specify the families $\mathcal{I}_i$, $\mathcal{O}_i$ since they will be clear from context. As an example if J is an $\mathcal{I}$-core then $\mathcal{I}[\{J\},\, \mathcal{I} - \{J\}]$, is the family of $\mathcal{I}$-sets that contain J and are disjoint from every other $\mathcal{I}$-core. It is useful to note that in general the family of complements of sets in $\mathcal{I}[\mathcal{A},\, \mathcal{B}]$ is $\mathcal{O}[\mathcal{B},\, \mathcal{A}]$.

Lemma 2.5 *For any color B with $\mathcal{O}_B \neq \emptyset$ and any set $O \in \mathcal{O}[\emptyset,\, \mathcal{O}_B]$, any $\mathcal{O}$-core is either contained in O or is disjoint from O.*

Proof: Let N be an $\mathcal{O}$-core. The $\mathcal{O}$-set O is not properly contained in the core N. If N and O are intersecting then they are crossing (since there are black $\mathcal{O}$-cores disjoint from O), a contradiction for core N. The remaining possibilities are as in the lemma. □

Suppose $\|\mathcal{I}\| \geq 2$. The *zone* $\widehat{J}$ of an $\mathcal{I}$-core J, also called an $\mathcal{I}$*-zone*, is the maximal set in $\mathcal{I}[\{J\},\, \mathcal{I} - \{J\}]$, i.e., the maximal $\mathcal{I}$-set that contains J and is disjoint from every other $\mathcal{I}$-core. It is easy to see that a unique zone $\widehat{J}$ exists since $\mathcal{I}$ is a crossing family.

When $\|\mathcal{I}\| > 2$ the $\mathcal{I}$-zones are pairwise disjoint. (If two zones $\widehat{I}$ and $\widehat{J}$ are not disjoint then they are intersecting. In fact they are crossing because of the third $\mathcal{I}$-core. This makes $\widehat{I} \cap \widehat{J}$ an $\mathcal{I}$-set containing an $\mathcal{I}$-core common to $\widehat{I}$ and $\widehat{J}$, a contradiction.)

An $\mathcal{I}$-core I *dominates* an $\mathcal{O}$-core O if $\widehat{O} \subseteq \widehat{I}$. It is possible for I and O to dominate each other. However there is no "dominance chain," i.e., I cannot dominate O and be dominated by a different $\mathcal{O}$-core N. This follows since such a chain implies $O \subseteq \widehat{I} \subseteq \widehat{N}$.

Lemma 2.6 *For an $\mathcal{I}$-core I and an $\mathcal{O}$-core O the following three conditions are equivalent:*
(i) $O \not\subseteq \widehat{I}$ and $I \not\subseteq \widehat{O}$;
(ii) O is disjoint from $\widehat{I}$ and I is disjoint from $\widehat{O}$;
(iii) neither of the two cores I, O dominates the other.

Proof: $(i) \implies (ii)$: If O is not disjoint from $\widehat{I}$ then Lemma 2.3 and $O \not\subseteq \widehat{I}$ show $\widehat{I} \subseteq O$, contradicting $I \not\subseteq \widehat{O}$.

$(ii) \implies (iii)$: Trivial.

$(iii) \implies (i)$: If $O \subseteq \widehat{I}$ then (iii) shows $\widehat{I}$ and $\widehat{O}$ are intersecting. Lemma 2.3 shows $\widehat{O} - \widehat{I}$ is an $\mathcal{O}$-set. But it contains no $\mathcal{O}$-core, a contradiction. □

2.2 Lower bound Φ

We will characterize OPT in terms of the quantity

$$\Phi = \max\{\|\mathcal{I}\|,\ \|\mathcal{O}\|,\ \|\mathcal{I}_B\| + \|\mathcal{O}_B\| : B \in \mathcal{P}\}.$$

We write $\Phi(\mathcal{F})$ if the family $\mathcal{F} = (\mathcal{I}, \mathcal{O})$ is unclear.

It is easy to see that $OPT \geq \Phi$: A solution must contain an edge entering every $\mathcal{I}$-core and an edge leaving every $\mathcal{O}$-core. Thus $OPT \geq \|\mathcal{I}\|, \|\mathcal{O}\|$. A single edge cannot both leave an $\mathcal{O}_B$-core and enter an $\mathcal{I}_B$-core. Thus $OPT \geq \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for any color B. Fig. 1(a) shows this lower bound need not be tight since it has $\Phi = 2$ and $OPT = 3$.

Fix a set of edges. $\mathcal{I}_u$ denotes the family of "uncovered" $\mathcal{I}$-sets, i.e., the $\mathcal{I}$-sets not covered by any edge of the set. Similarly we have $\mathcal{O}_u$ and $\mathcal{F}_u = (\mathcal{I}_u, \mathcal{O}_u)$.

$\mathcal{I}_u$ is a crossing family with $\|\mathcal{I}_u\| \leq \|\mathcal{I}\|$. To prove this it suffices to assume there is only one edge e (by induction). If a set $I \cup J$ or $I \cap J$ is covered by e then at least one of I and J is covered by e. Hence $\mathcal{I}_u$ is crossing. $\|\mathcal{I}_u\| \leq \|\mathcal{I}\|$ since $\mathcal{I}_u \subseteq \mathcal{I}$.

These observations justify the basic approach: We solve our covering problem by repeatedly adding a set of edges F to the desired cover. This leaves us with another covering problem on the family $\mathcal{F}_u$. We show each such addition decreases Φ by $|F|$. The following facts help establish this decrease.

Lemma 2.7 *For any set of edges, any set of $\mu(\mathcal{I}_u) - \mu(\mathcal{I})$ is mixed.*

Proof: Let $J \in \mu(\mathcal{I}_u) - \mu(\mathcal{I})$. J contains an $\mathcal{I}$-core that is covered by some edge e. Since J is not covered it contains both ends of e, making J mixed. □

$\mathcal{I}_{u,B}$ denotes the family of black sets in $\mathcal{I}_u$.

Lemma 2.8 *Suppose $\|\mathcal{I}\| \geq 2$. For any $\mathcal{I}$-core I, an edge from $V - \widehat{I}$ to I has $\|\mathcal{I}_u\| = \|\mathcal{I}\| - 1$. Furthermore if I is black then $\|\mathcal{I}_{u,B}\| = \|\mathcal{I}_B\| - 1$.*

Proof: Let e be the edge. Consider the first assertion. Since $\|\mathcal{I}\| \geq 2$ the $\mathcal{I}$-cores are pairwise disjoint. Hence every set of $\mu(\mathcal{I}) - \{I\}$ is an $\mathcal{I}_u$-core. Lemma 2.1 implies it suffices to show there is no $\mathcal{I}_u$-core disjoint from each set of $\mu(\mathcal{I}) - \{I\}$. Such a core contains I and is contained in $\widehat{I}$. Hence it is covered by e.

The second assertion is now obvious if $\|\mathcal{I}_u\| > 1$. If $\|\mathcal{I}_u\| = 1$ it follows from Lemma 2.7. □

The proof of the incrementation theorem eventually reduces any given family to one where $\|\mathcal{I}\|$ or $\|\mathcal{O}\|$ equals 1.

Lemma 2.9 *If* $\|\mathcal{I}\| = 1$ *then* $OPT = \Phi$.

Proof: We begin with an observation on the case $\|\mathcal{I}\| = 1$. Suppose there is a black $\mathcal{I}$-core I, and take any set of edges (even $\emptyset$). Then any $\mathcal{I}_u$-core distinct from I is mixed. This follows from Lemma 2.1 when there are no edges, and otherwise from Lemma 2.7.

We now prove the lemma by induction on Φ. First suppose the given family $\mathcal{F}$ has $\Phi = \|\mathcal{O}\| + 1$, i.e., there is a black $\mathcal{I}$-core I and $\|\mathcal{O}\|$ black $\mathcal{O}$-cores. Take an edge from a white element to I and let $\mathcal{F}_u = (\mathcal{I}_u, \mathcal{O}_u)$. Every $\mathcal{I}_u$-core is mixed by the preliminary observation. Hence $\Phi(\mathcal{F}_u) = \Phi(\mathcal{F}) - 1$ as desired.

Now we can assume $\Phi = \|\mathcal{O}\|$. Let I be an $\mathcal{I}$-core chosen to be mixed if possible. Let O be an $\mathcal{O}$-core chosen so that $I \cup O$ is mixed. Such an O exists since $\Phi = \|\mathcal{O}\|$.

Suppose $\|\mathcal{O}\| = 1$. Take an edge from O to I. (There is such an edge respecting the bipartition since $I \cup O$ is mixed.) Suppose N is an $\mathcal{O}_u$-set. $N \not\subseteq I$ since that would make N and $\overline{I}$ disjoint $\mathcal{O}$-sets. Similarly $I \not\subseteq N$. Hence Lemma 2.3 shows N is disjoint from I. Corollary 2.2 shows O is the unique $\mathcal{O}$-core, so $O \subseteq N$. Thus N is covered, $\mathcal{O}_u = \emptyset$ and $OPT = 1$ as desired.

Suppose $\|\mathcal{O}\| > 1$. Let N be an $\mathcal{O}$-core distinct from O. We first show that $N \cup (V - \widehat{N})$ is mixed. If not we can say it is a black set. Then there is a black $\mathcal{I}$-core contained in the $\mathcal{I}$-set $V - \widehat{N}$. All $\mathcal{O}$-cores other than N are in this set as well and so are black. Thus $\Phi = \|\mathcal{O}\| + 1$, a contradiction.

Hence we can choose an element $x \notin \widehat{N}$ so that $N \cup x$ is mixed. Take an edge from N to x. Lemma 2.8 shows $\|\mathcal{O}_u\| = \|\mathcal{O}\| - 1$. Our choice of I and O, plus the preliminary observation and Lemma 2.7, shows $\Phi(\mathcal{F}_u) = \|\mathcal{O}_u\|$ as desired. $\square$

2.3 Compatible pairs

This section discusses the main tool of our reduction, compatible pairs. From now on assume $\|\mathcal{I}\|, \|\mathcal{O}\| > 1$ since otherwise we are done by Lemma 2.9.

An $\mathcal{I}$-core I and an $\mathcal{O}$-core O form a *compatible pair* if $I \cup O$ is mixed and neither core dominates the other. (We denote such a compatible pair as I, O or O, I depending on convenience.) To *augment* such a compatible pair O, I means to place in the cover an edge directed from an element of O to a differently colored element of I. The *augmentation* of O, I is the corresponding family $\mathcal{F}_u$. Lemma 2.8 implies the augmentation has a lower value of $\|\mathcal{I}\|$ and $\|\mathcal{O}\|$, and a lower value of $\|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ if $O \in \mathcal{O}_B$ or $I \in \mathcal{I}_B$. Thus an augmentation has a lower value of Φ if O and I are of opposite colors, i.e., neither O nor I is mixed. (This condition is not necessary.)

Lemma 2.10 *Suppose* $\|\mathcal{I}\| + \|\mathcal{O}\| \geq 2\Phi - 1$. *If* I_i, $i = 1, 2$ *are distinct* $\mathcal{I}$*-cores and* O_j, $j = 1, 2$ *are distinct* $\mathcal{O}$*-cores with each of the four sets* $I_i \cup O_j$ *mixed, then some* I_i, O_j *is compatible.*

Proof: Suppose there is no compatible pair I_i, O_j. Since I_1 is incompatible with O_1 we can assume without loss of generality that I_1 dominates O_1. This implies that I_1 and I_2 both dominate O_1 and O_2 (since there is no dominance chain). We conclude $\widehat{I_1}$ and $\widehat{I_2}$ are intersecting zones. Thus $\|\mathcal{I}\| = 2$. Furthermore $\widehat{I_1}$ and $\widehat{I_2}$ are co-disjoint. Each $\mathcal{O}$-set $V - \widehat{I_i}$ contains an $\mathcal{O}$-core N_i. Since N_i is distinct from $O_1, O_2 \subseteq \widehat{I_1} \cap \widehat{I_2}$ we get the contradiction $\Phi \geq \|\mathcal{O}\| \geq \|\mathcal{I}\| + 2$. $\square$

Corollary 2.11 *Suppose* $\|\mathcal{I}\| + \|\mathcal{O}\| \geq 2\Phi - 1$. *If each* $\mathcal{I}$*-core and each* $\mathcal{O}$*-core have a mixed union then* $OPT = \Phi$.

Proof: Assume $\|\mathcal{I}\| \geq \|\mathcal{O}\| > 1$ by Lemma 2.9. Clearly $\Phi = \|\mathcal{I}\|$. Lemma 2.10 shows there is a compatible pair of cores. Let $\mathcal{F}_u$ be the augmentation. If $\|\mathcal{O}_u\| > 1$ then clearly $\Phi(\mathcal{F}_u) = \Phi(\mathcal{F}) - 1$, $\mathcal{F}_u$ satisfies the lemma's hypothesis and we are done by induction. If $\|\mathcal{O}_u\| = 1$ Lemma 2.7 implies $\Phi(\mathcal{F}_u) = \Phi(\mathcal{F}) - 1$. Hence we are done by Lemma 2.9. $\square$

2.4 Crossing family operations

We perform several operations that modify one crossing family into another. Let $\mathcal{F}$ be a crossing family over a set of elements V. In what follows assume x is an arbitrary element of V.

Let S be the set $\{x\}$ or $V - \{x\}$. *Making S an $\mathcal{F}$-set* means to enlarge the family with the set S (if it is not already present). This gives another crossing family for our choices of S, since $\{x\}$ and $V - x$ are not crossing with any set. Note that making $\{x\}$ an $\mathcal{O}$-set automatically makes $V - x$ an $\mathcal{I}$-set.

Deleting x means to delete x from V and from any $\mathcal{F}$-set in which it occurs. This gives a new crossing family since sets that are crossing in the new family are crossing in the original family.

Duplicating x means adding a new element x' to V with x' belonging to exactly the same sets as x. Thus a set S in the original family corresponds to a set S' in the new family with $S' = S$ if $x \notin S$ and $S' = S \cup x'$ if $x \in S$. The new family is crossing since again sets that are crossing in the new family are crossing in the original family.

3 Singleton families

We will reduce our covering problem to the case where every $\mathcal{I}$- and $\mathcal{O}$-core is a singleton set. We call such an $\mathcal{F}$ a *singleton family*. Throughout this section assume the given family $\mathcal{F}$ is a singleton family. We typically denote an $\mathcal{I}$-core by x, where $x \in V$ and we have identified x and $\{x\}$ as usual. Similarly an $\mathcal{O}$-core is typically denoted by y. The notational conventions also imply that expressions such as $\mathcal{I}_B - \{x\}$ and $\mathcal{I}_B \cup \{x, y\}$ are shorthands for $\mathcal{I}_B - \{\{x\}\}$ and $\mathcal{I}_B \cup \{\{x\}, \{y\}\}$ respectively.

Note that augmenting a compatible pair in a singleton family decreases Φ. This follows from Lemma 2.8. The augmentation is a singleton family by Lemma 2.1.

A *1-blocker (for B)* is a singleton family with $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ and $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$. Fig. 1(a) is a 1-blocker. The next section shows that such a 1-blocker has $OPT > \Phi$. Note that $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$ if and only if $\mathcal{I}[\mathcal{O}_B, \mathcal{I}_B] \neq \emptyset$. Hence $\mathcal{F}$ is a 1-blocker if and only if its complement-symmetric family is.

A 1-blocker has an $\mathcal{I}$-core and an $\mathcal{O}$-core of both colors, i.e., $\mathcal{I}_B, \mathcal{O}_B, \mathcal{I}_W, \mathcal{O}_W \neq \emptyset$. In proof, a set of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ contains a white $\mathcal{O}$-core and its complement contains a white $\mathcal{I}$-core. If $\|\mathcal{I}_B\| = 0$ then we get the contradiction $\Phi = \|\mathcal{O}_B\| < \|\mathcal{O}\|$.

Let $\mathcal{F}$ be a singleton family with $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ that is not a 1-blocker for B. An $\mathcal{I}_B$-core x is *dangerous* if $\mathcal{I}_B - \{x\},\ \mathcal{O}_B,\ \mathcal{O}[\mathcal{I}_B - \{x\}, \mathcal{O}_B] \neq \emptyset$. This definition implies $\mu^*(\mathcal{O}[\mathcal{I}_B - \{x\}, \mathcal{O}_B])$ is well-defined (since $\mathcal{O}$ is crossing). We also have the complement-symmetric definition, an $\mathcal{O}_B$-core y is *dangerous* if $\mathcal{O}_B - \{y\},\ \mathcal{I}_B,\ \mathcal{I}[\mathcal{O}_B - \{y\}, \mathcal{I}_B] \neq \emptyset$.

If x is dangerous then it does not belong to any set of $\mathcal{O}[\mathcal{I}_B - \{x\}, \mathcal{O}_B]$. If it did then $\mathcal{F}$ would be a 1-blocker.

Lemma 3.1 *Let $\mathcal{F}$ be a singleton family. Let $\mathcal{F}_u$ be the augmentation of a compatible pair x, y where x is an $\mathcal{I}_B$-core. Suppose $\mathcal{F}$ is not a 1-blocker for B but $\mathcal{F}_u$ is. Then x is dangerous. Furthermore $\mathcal{O}[\mathcal{I}_B - \{x\}, \mathcal{O}_B \cup \{x, y\}] \neq \emptyset$.*

Proof: Let $\mathcal{F}_u = (\mathcal{I}_u, \mathcal{O}_u)$. The black cores are the same in $\mathcal{F}$ and $\mathcal{F}_u$ except that x is not an $\mathcal{I}_u$-core. Hence $\mathcal{F}_u$ a 1-blocker for B implies that in $\mathcal{F}$, $\mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B] \neq \emptyset$. It also implies $\mathcal{F}$ has $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$. There is a black $\mathcal{I}_u$-core and a black $\mathcal{O}_u$-core. Thus x is dangerous.

Take any uncovered set $N \in \mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B]$. As shown above N does not include x. N does not include y since N is uncovered. This gives the second part of the lemma. □

Lemma 3.2 *If $\|\mathcal{I}_B\| > 2$ then at most one $\mathcal{I}_B$-core is dangerous.*

Proof: Suppose x_i, $i = 1, 2$ are two dangerous black $\mathcal{I}$-cores, with corresponding sets $O_i \in \mathcal{O}[\mathcal{I}_B - \{x_i\},\, \mathcal{O}_B \cup \{x_i\}]$. The hypothesized third black $\mathcal{I}$-core shows O_1 and O_2 are intersecting. They are crossing since $\mathcal{O}_B \neq \emptyset$. This makes $O_1 \cup O_2$ an $\mathcal{O}$-set and $\mathcal{F}$ a 1-blocker, a contradiction. □

Lemma 3.3 *Let x be a dangerous $\mathcal{I}_B$-core. Let y be any $\mathcal{O}$-core contained in $\mu^*(\mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B])$. Then x, y is a compatible pair whose augmentation is not a 1-blocker for B.*

Proof: Let $\mu^*(\mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B \cup \{x\}]) = O$. Clearly O contains an $\mathcal{O}$-core y and any such y is white.

Suppose $x \in \widehat{y}$. $\mathcal{F}$ has a black $\mathcal{O}$-core, which is disjoint from $O \cup \widehat{y}$. Thus either $O \subseteq \widehat{y}$ or O and $\widehat{y}$ are crossing. In both cases $O \cup \widehat{y}$ is an $\mathcal{O}$-set, and it contains x. But this makes $\mathcal{F}$ a 1-blocker for B. We conclude $x \notin \widehat{y}$.

Suppose $y \in \widehat{x}$. Since O contains an $\mathcal{I}$-core distinct from x, O and $\widehat{x}$ are intersecting. Lemma 2.3 makes $O - \widehat{x}$ an $\mathcal{O}$-set that contradicts the minimality of O. We conclude $y \notin \widehat{x}$.

Lemma 2.6 shows that x, y is a compatible pair. If its augmentation is a 1-blocker for B then Lemma 3.1 shows $\mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B \cup \{x, y\}] \neq \emptyset$. This again contradicts the minimality of O. □

Suppose a singleton family is not a 1-blocker (for either color). Let x, y be a compatible pair with x an $\mathcal{I}_B$-core and y an $\mathcal{O}_W$-core. x, y is *safe* if $y \in \mu^*(\mathcal{O}[\mathcal{I}_B - \{x\},\, \mathcal{O}_B])$ when x is dangerous and $x \in \mu^*(\mathcal{I}[\mathcal{O}_W - \{y\},\, \mathcal{I}_W])$ when y is dangerous. The augmentation of a safe pair is not a 1-blocker by Lemmas 3.1 and 3.3 (applied to both x and y).

Lemma 3.4 *Let $\mathcal{F}$ be a singleton family. For $i = 1, 2$, let x_i, y_i be compatible pairs for distinct $\mathcal{I}$-cores x_i, $i = 1, 2$ and distinct $\mathcal{O}$-cores y_i, $i = 1, 2$. Augmenting both pairs decreases $\|I\|$ by 2 if and only if $\mathcal{I}[\{x_1, x_2, y_1, y_2\},\, \mathcal{I} - \{x_1, x_2\}] = \emptyset$.*

Proof: Let $\mathcal{F}_u$ be the augmentation. For the only if direction suppose $\mathcal{I}[\{x_1, x_2, y_1, y_2\},\, \mathcal{I} - \{x_1, x_2\}] \neq \emptyset$. A minimal set in this family is an $\mathcal{I}_u$-core that shows $\|\mathcal{I}\|$ does not decrease by 2. This proves the only if direction.

For the if direction it suffices to show there is no $\mathcal{I}_u$-set K that contains no $\mathcal{I}$-core other than x_1 or x_2. Without loss of generality $x_1 \in K$. We must also have $x_2 \in K$ since otherwise $K \subseteq \widehat{x_1}$ and K is covered. We must have $y_1, y_2 \in K$, else again K is covered. So $K \in \mathcal{I}[\{x_1, x_2, y_1, y_2\},\, \mathcal{I} - \{x_1, x_2\}] = \emptyset$, a contradiction. □

For completeness we note that this lemma need not hold for a general family $\mathcal{F}$ (rather than a singleton family) unless $\|\mathcal{I}\| \geq 4$.

We use the following operation on a singleton family $\mathcal{F}$: To *create an $\mathcal{I}_B$-core*, start by choosing a black element x, where if possible x is an $\mathcal{O}_B$-core. If x is not an $\mathcal{I}$-core then simply make $\{x\}$ an $\mathcal{I}$-set. If x is an $\mathcal{I}$-core then proceed as follows. First duplicate x (in the crossing family $\mathcal{I}$). Recall from Section 2.4 that this gives a new element x'. $\{x, x'\}$ is now an $\mathcal{I}$-core. Also it is an $\mathcal{O}$-core if x is an $\mathcal{O}$-core of the original family $\mathcal{F}$ (since this means that $V-x$ is an $\mathcal{I}$-set of the original family). Add x' to B. Make $\{x\}$ and $\{x'\}$ $\mathcal{I}$-sets. If x is an $\mathcal{O}$-core of $\mathcal{F}$ then make $\{x\}$ (but not $\{x'\}$) an $\mathcal{O}$-set.

Suppose $\mathcal{F}$ is the original family and $\mathcal{F}'$ the new family. Assume x is not the unique $\mathcal{O}$-core of $\mathcal{F}$. This operation increases $\|\mathcal{I}\|$ and $\|\mathcal{I}_B\|$ by 1. The assumption implies it does not change the $\mathcal{O}$-cores (even when $V-x$ becomes a new $\mathcal{O}$-set) and also that $\mathcal{F}'$ is a singleton family. Clearly a cover of $\mathcal{F}'$ gives a cover of $\mathcal{F}$ with the same number of edges.

$\mathcal{F}'$ is not a 1-blocker for B (since the choice of x implies $\mathcal{F}'$ has either $\mathcal{O}_B=\emptyset$ or $\mathcal{O}[\mathcal{I}_B,\mathcal{O}_B]=\emptyset$.) Furthermore $\mathcal{F}'$ is a 1-blocker (for W) only if $\mathcal{F}$ is. To see this, $\mathcal{F}'$ a 1-blocker for W obviously implies Φ does not increase. The complement of a set in $\mathcal{O}[\mathcal{I}_W,\mathcal{O}_W]$ is in $\mathcal{I}[\mathcal{O}_W,\mathcal{I}_W]$ and is easily seen not to be a newly created $\mathcal{I}$-set (which is either $\{x\}$, $\{x'\}$ or $V\cup x'-x$).

Lemma 3.5 *A singleton family that is not a 1-blocker has $OPT=\Phi$.*

Proof: Assume $\mathcal{F}$ is not a 1-blocker. Assume $\|\mathcal{I}\|,\|\mathcal{O}\|>1$ by Lemma 2.9.

We first modify $\mathcal{F}$ to make

$$\Phi=\|\mathcal{I}_B\|+\|\mathcal{O}_B\|=\|\mathcal{I}_W\|+\|\mathcal{O}_W\|. \tag{1}$$

Notice that a family having three of the four quantities $\|\mathcal{I}\|$, $\|\mathcal{O}\|$, $\|\mathcal{I}_B\|+\|\mathcal{O}_B\|$, $\|\mathcal{I}_W\|+\|\mathcal{O}_W\|$ equal actually has all four quantities equal and satisfies (1). This follows since the number of cores is $\|\mathcal{I}\|+\|\mathcal{O}\|=(\|\mathcal{I}_B\|+\|\mathcal{O}_B\|)+(\|\mathcal{I}_W\|+\|\mathcal{O}_W\|)$. To achieve (1) when $\Phi=\|\mathcal{I}_B\|+\|\mathcal{O}_B\|$ for some color B, create $(\Phi-\|\mathcal{I}\|)$ $\mathcal{I}_W$-cores and $(\Phi-\|\mathcal{O}\|)$ $\mathcal{O}_W$-cores. (We now have $\|\mathcal{I}_B\|+\|\mathcal{O}_B\|=\|\mathcal{I}\|=\|\mathcal{O}\|$.) In the opposite case without loss of generality assume $\Phi=\|\mathcal{O}\|$. Create $\Phi-(\|\mathcal{I}_B\|+\|\mathcal{O}_B\|)$ $\mathcal{I}_B$-cores and $\Phi-(\|\mathcal{I}_W\|+\|\mathcal{O}_W\|)$ $\mathcal{I}_W$-cores. (We now have $\|\mathcal{O}\|=\|\mathcal{I}_B\|+\|\mathcal{O}_B\|=\|\mathcal{I}_W\|+\|\mathcal{O}_W\|$.) In both cases the new family satisfies (1), has the same value of Φ and is not a 1-blocker. So it suffices to show the new family has $OPT=\Phi$. From now on we refer to the new family as $\mathcal{F}$.

(1) implies $\|\mathcal{I}_B\|=\|\mathcal{O}_W\|$ and $\|\mathcal{I}_W\|=\|\mathcal{O}_B\|$. In proof $\|\mathcal{I}_B\|+\|\mathcal{O}_B\|=\Phi\ge\|\mathcal{I}\|$ implies $\|\mathcal{O}_B\|\ge\|\mathcal{I}_W\|$. Similarly $\|\mathcal{I}_B\|\ge\|\mathcal{O}_W\|$. Now (1) implies both these inequalities hold with equality.

We prove $OPT=\Phi$ by induction on Φ. It suffices to find a safe pair (since the augmentation $\mathcal{F}_u$ is not a 1-blocker and $\Phi(\mathcal{F}_u)=\Phi(\mathcal{F})-1$).

Observe that if there are two $\mathcal{I}_B$-cores and two $\mathcal{O}_W$-cores, none of which are dangerous, Lemma 2.10 gives a compatible pair which is clearly safe. This observation gives the desired safe pair if $\|\mathcal{I}_B\|>2$, since Lemma 3.2 shows at most one $\mathcal{I}_B$-core and at most one $\mathcal{O}_W$-core are dangerous. We also get the desired safe pair if $\|\mathcal{I}_B\|=0$ since then no core is dangerous. Suppose $\|\mathcal{I}_B\|=1$. The unique $\mathcal{O}_W$-core y and the unique $\mathcal{I}_B$-core x form a compatible pair y,x. This follows since $x\in\widehat{y}$ makes $\mathcal{F}$ a 1-blocker for B and $y\in\widehat{x}$ makes $\mathcal{F}$ a 1-blocker for W. y,x is the desired safe pair since neither x nor y is dangerous by definition.

The remaining possibility is $\|\mathcal{I}_B\|=2$. Hence assume

$$I_B=\{x_1,x_2\},\quad O_W=\{y_1,y_2\}.$$

The observation shows we can assume at least one of these four cores is dangerous, say x_1. If no $\mathcal{O}_W$-core is dangerous then Lemma 3.3 gives a safe pair. Hence without loss of generality assume y_1 is dangerous. (We remark that in addition either or both of y_2 and x_2 can be dangerous.)

Write $\mu^*(\mathcal{O}[\mathcal{I}_B-\{x_1\},\mathcal{O}_B\cup\{x_1\}])=O$. By symmetry, the definition of safe and Lemma 3.3 we can assume $y_1\notin O$.

Claim 1: Both y_1,x_2 and y_2,x_1 are compatible.

Proof: $y_2 \in O$ since O contains an $\mathcal{O}$-core. Thus

$$O \in \mathcal{O}[\{x_2, y_2\},\, \mathcal{O}_B \cup \{x_1, y_1\}].$$

In particular Lemma 3.3 shows y_2, x_1 is compatible.

Claim 1 follows if y_1, x_2 is compatible so assume not. In particular writing $\mu^*(\mathcal{I}[\mathcal{O}_W - \{y_1\},\, \mathcal{I}_W \cup \{y_1\}]) = I$, Lemma 3.3 implies $x_2 \notin I$. $x_1 \in I$ since I contains an $\mathcal{I}$-core. Thus

$$I \in \mathcal{I}[\{x_1, y_2\},\, \mathcal{I}_W \cup \{x_2, y_1\}].$$

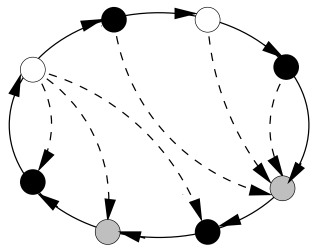
We have shown O and I are intersecting: $y_2 \in O \cap I$, $x_2 \in O - I$, $x_1 \in I - O$. Lemma 2.3 shows $O - I$ is an $\mathcal{O}$-set. y_2 is the only $\mathcal{O}$-core in O, so $O - I$ contains no $\mathcal{O}$-core – a contradiction. ♠

Now augment both compatible pairs of Claim 1. Note that $\mathcal{I}[\{x_1, x_2, y_1, y_2\},\, \mathcal{I} - \{x_1, x_2\}] = \emptyset$, since otherwise $\mathcal{F}$ is a 1-blocker for W. So Lemma 3.4 shows $\|I\|$ decreases by 2. Similarly $\|O\|$ decreases by 2. Thus Φ decreases by 2. The family $\mathcal{F}_u$ is not a 1-blocker since it has no $\mathcal{I}_B$-core. Hence we are done by induction. □

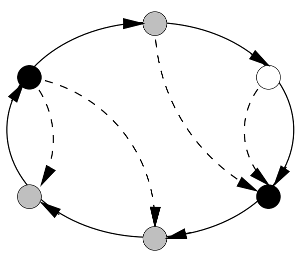
4 Blockers

In general there are six types of blockers, illustrated in Fig. 2 for our edge-connectivity incrementation problem. We begin by defining the blockers, and then we discuss Fig. 2 in detail.

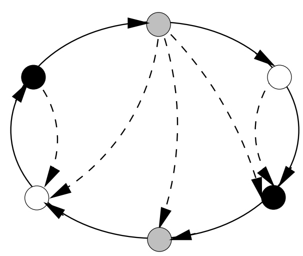
A family $\mathcal{F}$ is a *blocker* if it satisfies any of the following six sets of conditions. A *1-blocker* has
  (i) $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$;
  (ii) $\mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_B] \neq \emptyset$.
The next three blockers all have the same first condition,
  (i) $\Phi = \|\mathcal{I}\| = \|\mathcal{O}\|$.
A *2-blocker* has
  (ii) $\mathcal{O}[\mathcal{I}_{B'},\, \mathcal{O}_{B'}] \neq \emptyset$.
A *3-blocker* is either a $3\,\mathcal{O}$-blocker or a $3\,\mathcal{I}$-blocker. A $3\,\mathcal{O}$*-blocker* has
  (ii) $\mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}],\ \mathcal{O}[\mathcal{I}_W,\, \mathcal{O}_{B'}] \neq \emptyset$;
  (iii) $\|\mathcal{I}_X\| = 1$.
A $3\,\mathcal{I}$*-blocker* satisfies the complement-symmetric conditions,
  (ii) $\mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}],\ \mathcal{I}[\mathcal{O}_W,\, \mathcal{I}_{B'}] \neq \emptyset$;
  (iii) $\|\mathcal{O}_X\| = 1$.
A *4-blocker* has
  (ii) $\mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}],\ \mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}] \neq \emptyset$;
  (iii) $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1$.
The last two blockers have the same first condition,
  (i) $\|\mathcal{I}\| + \|\mathcal{O}\| = 2\Phi - 1$.
Thus either $\|\mathcal{I}\| = \Phi$, $\|\mathcal{O}\| = \Phi - 1$ or $\|\mathcal{O}\| = \Phi$, $\|\mathcal{I}\| = \Phi - 1$. A *5-blocker* is either a $5\,\mathcal{O}$-blocker or a $5\,\mathcal{I}$-blocker. A $5\,\mathcal{O}$*-blocker* has
  (ii) $\mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}],\ \mathcal{O}[\mathcal{I}_W,\, \mathcal{O}_{B'}] \neq \emptyset$;
  (iii) $\|\mathcal{I}_X\| = 0$.
A $5\,\mathcal{I}$*-blocker* satisfies the complement-symmetric conditions,
  (ii) $\mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}],\ \mathcal{I}[\mathcal{O}_W,\, \mathcal{I}_{B'}] \neq \emptyset$;



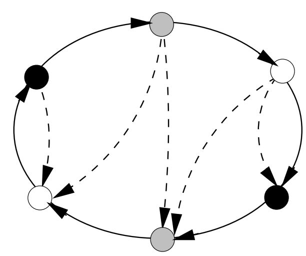
(a) 1-blocker



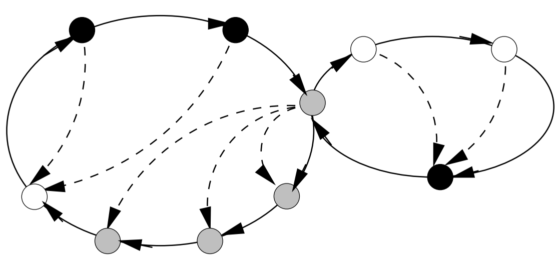
(b) 2-blocker



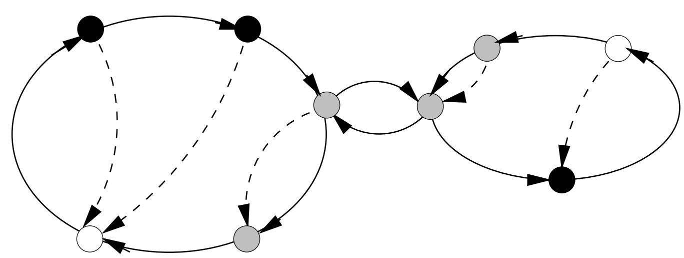
(c) 3 $\mathcal{O}$-blocker



(d) 4-blocker



(e) 5 $\mathcal{O}$-blocker



(f) 6-blocker

Figure 2: Blockers for edge-connectivity incrementation. A directed line represents an edge that has multiplicity k if it is drawn solid, multiplicity 1 if it is dashed. (a) and (f) have $\Phi = 4$, (b) – (d) have $\Phi = 3$, (e) has $\Phi = 5$. Sets showing condition (ii) is satisfied are (a) $\{2,3,4,5\}$ (b) $\{2,3,4\}$ (c) and (d) $\{6,1\}$, $\{3,4\}$ (e) $\{7,1,2\}$ in the left ring, $\{1,2,3\}$ in the right ring, (f) $\{5,1,2\}$ in the left ring, $\{4,1\}$, $\{1,2,3,4\}$ in the right ring.

(*iii*) $\|\mathcal{O}_X\| = 0$.

A 6-*blocker* has

(*ii*) $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}],\ \mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}],\ \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}] \neq \emptyset$;

(*iii*) $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1$.

The colors B and W are symmetric for 3- and 5-blockers. For the other blockers we say *blocker for* B or *blocker for* W to specify the color that plays the main role in the conditions. The above definitions all specify blockers for B. A 1-blocker for W has $\Phi = \|\mathcal{I}_W\| + \|\mathcal{O}_W\|$ and $\mathcal{O}[\mathcal{I}_W, \mathcal{O}_W] \neq \emptyset$. In a 4-blocker for W condition (*iii*) is $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| = \Phi - 1$.

Now consider the examples of Fig. 2. Each graph is k-edge-connected and we wish to increase the edge-connectivity by 1. Here k is an arbitrary positive integer. (The case $k = 0$ is discussed separately below.) It is useful to observe that the solid edges of each graph form 1, 2 or 3 simple cycles, that we refer to as "rings." These rings form a k-edge-connected spanning subgraph. Hence no dashed edge can leave an $\mathcal{O}$-set or enter an $\mathcal{I}$-set.

Every core in each graph is a singleton. In each graph the $\mathcal{I}$-cores are drawn on the top and the $\mathcal{O}$-cores are drawn on the bottom. Three vertices are not cores – they are the shaded vertices drawn in the middle of (e) and (f). A black or white vertex represents a vertex of that color. A shaded vertex represents a mixed core – it can be replaced by a black vertex and a white vertex joined to each other by two antiparallel edges of multiplicity $k + 1$.

We number the vertices of each ring consecutively, starting with the $\mathcal{I}$-core whose immediate predecessor in the ring is not an $\mathcal{I}$-core. So for instance in Fig. 2(e) the $\mathcal{O}$-cores are vertices 4 through 7 of the left ring and vertex 3 of the right ring. Note that the vertices of any ring that belong to a given $\mathcal{O}$-set or $\mathcal{I}$-set have consecutive numbers assuming we allow wrap-around, e.g., in Fig. 2(e) $\{7, 1, 2\}$ in the left ring is an $\mathcal{I}$-set and an $\mathcal{O}$-set.

The six types of blockers are distinct. For instance each blocker in Fig. 2 satisfies only one set of blocker conditions. If we ignore symmetries there are actually 10 varieties of blockers: A $3\,\mathcal{I}$-blocker is illustrated by reversing the edges in Fig. 2(c). A $5\,\mathcal{O}$-blocker with $\|\mathcal{I}\| > \|\mathcal{O}\|$ is illustrated by adding two $\mathcal{I}$-cores to Fig. 2(e), copying an $\mathcal{I}_B$-core in the left ring and an $\mathcal{I}_W$-core in the right ring. $5\,\mathcal{I}$-blockers are illustrated by reversing the edges of these two $5\,\mathcal{O}$-blockers. Again each of these 10 graphs satisfies just one set of blocker conditions.

A slight modification of the figure shows that all 10 varieties of blockers occur even for bipartite digraph edge-connectivity incrementation (i.e., the framework reinforcement problem). The graphs in Fig. 2(a)-(d) are bipartite, assuming edges incident to mixed vertices are handled appropriately. Fig. 2(e)-(f) have 3 (solid) edges that violate bipartiteness (edge $1, 2$ in the left and right rings of (e) and in the left ring of (f)). In each case the edge xy joins two vertices (x and y) that are tails of dashed edges directed to the same vertex z. Replace edge xy by a new mixed vertex w, solid edges xw, wy and dashed edges xw, wz. The new graph is bipartite, has the same cores and is a blocker of the same type.

There are simpler blockers than those in Fig. 2. For example Fig. 1(a) shows a 1- and 2-blocker for both B and W. The graph formed from Fig. 2(e) by getting rid of one black $\mathcal{I}$-core and two mixed $\mathcal{O}$-cores from the left ring and one white $\mathcal{I}$-core from the right ring is a $5\,\mathcal{O}$-blocker, and a 6-blocker for both B and W. The slightly larger graphs of Fig. 2 were chosen so that each graph satisfies only one set of blocker conditions.

If $k = 0$ the graphs of Fig. 2(a)–(d) are 1-, 2-, 3- and 4-blockers respectively. (These graphs are essentially the same as the examples in [8].) The graph of Fig. 2(e) is not a 5-blocker but rather a $3\,\mathcal{O}$-blocker. The graph of Fig. 2(f) is not a blocker at all. See Lemma 6.1 below.

Lemma 4.1 *Any blocker has* $OPT > \Phi$.

Proof: We consider each type of blocker separately. For each blocker we assume that $OPT = \Phi$ and derive a contradiction.

1-blocker. At least $\|\mathcal{I}_B\|$ edges enter a black $\mathcal{I}$-core and at least $\|\mathcal{O}_B\|$ other edges leave a black $\mathcal{O}$-core. Hence condition (i) implies every edge is incident to a black core. But a set $O \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ cannot be covered by such an edge, since O does not contain an element of a black $\mathcal{O}$-core and $\overline{O}$ does not contain an element of a black $\mathcal{I}$-core.

2-blocker. Condition (i) shows every edge leaves an $\mathcal{O}$-core and enters an $\mathcal{I}$-core. But a set $O \in \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$ cannot be covered by such an edge, since the only $\mathcal{O}$-cores with an element in O are black and the only $\mathcal{I}$-cores with an element in $\overline{O}$ are black.

3-blocker. Consider a $3\,\mathcal{O}$-blocker. Condition (i) shows every edge leaves an $\mathcal{O}$-core and enters an $\mathcal{I}$-core. An edge covering a set of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$ leaves a white $\mathcal{O}$-core and enters a mixed $\mathcal{I}$-core. Similarly an edge covering a set of $\mathcal{O}[\mathcal{I}_W, \mathcal{O}_{B'}]$ leaves a black $\mathcal{O}$-core and enters a mixed $\mathcal{I}$-core. These two edges are distinct and they enter the same mixed $\mathcal{I}$-core (condition (iii)). But (i) implies no two edges enter the same $\mathcal{I}$-core.

$3\,\mathcal{I}$-blockers are handled by complement-symmetry.

4-blocker. Condition (i) shows every edge leaves an $\mathcal{O}$-core and enters an $\mathcal{I}$-core. An edge covering a set of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$ leaves a white $\mathcal{O}$-core and enters a mixed $\mathcal{I}$-core. Similarly an edge covering a set of $\mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$ enters a white $\mathcal{I}$-core and leaves a mixed $\mathcal{O}$-core. These two edges are distinct and neither covers a black core, contradicting condition (iii).

5-blocker. Consider a $5\,\mathcal{O}$-blocker. There are $2\Phi - 1$ cores in the families of cores corresponding to $\|\mathcal{I}\|$ and $\|\mathcal{O}\|$. Hence $\Phi - 1$ edges cover two cores and one "exceptional" edge covers (at least) one core. An edge e covering a set in $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$ can leave a white $\mathcal{O}$-core or enter a white $\mathcal{I}$-core (condition (iii) shows there are no mixed $\mathcal{I}$-cores). Since e cannot do both, e must be the exceptional edge. Similarly an edge f covering a set in $\mathcal{O}[\mathcal{I}_W, \mathcal{O}_{B'}]$ can leave a black $\mathcal{O}$-core or enter a black $\mathcal{I}$-core, so f must be the exceptional edge. Clearly $e \neq f$, a contradiction.

$5\,\mathcal{I}$-blockers are handled by complement-symmetry.

6-blocker. A 6-blocker is complement-symmetric, since $\mathcal{I}[\mathcal{O}_{B'}, \mathcal{I}_{B'}]$ is the set of all complements of sets of $\mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$. Hence without loss of generality we can assume $\|\mathcal{O}\| = \Phi$.

The assumption implies every edge leaves an $\mathcal{O}$-core. To cover a set in $\mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$ some edge leaves a black $\mathcal{O}$-core and does not enter an $\mathcal{I}$-core. This implies every other edge enters an $\mathcal{I}$-core. To cover a set in $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$ some edge leaves a white $\mathcal{O}$-core and does not enter a black $\mathcal{I}$-core. To cover a set in $\mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$ some edge enters a white $\mathcal{I}$-core and does not leave a black $\mathcal{O}$-core. This gives two distinct edges that are not incident to black cores, contradicting (iii). $\square$

For a blocker type i (i.e., $i \in \{1, 2, 3\mathcal{I}, 3\,\mathcal{O}, 4, 5\mathcal{I}, 5\,\mathcal{O}, 6\}$) a *nondegenerate i-blocker* has every family of cores implicitly referred to in condition (ii) nonempty. For instance a nondegenerate 2-blocker has a nonblack $\mathcal{I}$-core and a nonblack $\mathcal{O}$-core; a nondegenerate $3\,\mathcal{O}$-blocker has a black and a white $\mathcal{I}$-core and a nonblack and a nonwhite $\mathcal{O}$-core. A blocker is *nondegenerate* if it is a nondegenerate i-blocker for some type i.

Lemma 4.2 *Any blocker is nondegenerate.*

Proof: Assume $\|\mathcal{I}\|, \|\mathcal{O}\| > 1$ by Lemmas 4.1 and 2.9.

1-blocker. By complement symmetry we need only prove there is a black $\mathcal{I}$-core. Condition (ii) implies there is a nonblack $\mathcal{O}$-core. Thus $\Phi \geq \|\mathcal{O}\| > \|\mathcal{O}_B\|$. Now condition (i) implies $\|\mathcal{I}_B\| > 0$ as desired.

2-blocker. By complement symmetry we need only prove there is a nonblack $\mathcal{I}$-core. Condition (ii) implies there is a black $\mathcal{O}$-core. Thus $\Phi \geq \|\mathcal{I}_B\| + \|\mathcal{O}_B\| > \|\mathcal{I}_B\|$. Now condition (i) implies $\|\mathcal{I}_{B'}\| > 0$ as desired.

3-blocker. By complement symmetry it suffices to consider a $3\,\mathcal{O}$-blocker. Condition (ii) implies there is a white and a black $\mathcal{O}$-core. Since B and W are symmetric it now suffices to show there is a black $\mathcal{I}$-core. We can assume the family is not a 1-blocker for W, by the previous case. Hence condition (ii) implies $\Phi > \|\mathcal{I}_W\| + \|\mathcal{O}_W\| \geq \|\mathcal{I}_W\| + 1$. Now conditions (iii) and (i) imply $\|\mathcal{I}\| > \|\mathcal{I}_{B'}\|$. Thus $\|\mathcal{I}_B\| > 0$ as desired.

4-blocker. We will show there is a black $\mathcal{I}$-core. Then complement symmetry implies there is a black $\mathcal{O}$-core, and these two facts imply the blocker is nondegenerate.

The first part of condition (ii) shows there is a white $\mathcal{O}$-core. Now we show there is also a mixed $\mathcal{O}$-core. We can assume the family is not a 2-blocker for W. Since $\mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B]$ is the family of complements of sets in $\mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$, the second part of condition (ii) implies $\mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B] \neq \emptyset$. This and the assumption imply there is a mixed $\mathcal{O}$-core.

We have shown $\Phi \geq \|\mathcal{O}\| \geq \|\mathcal{O}_B\| + 2$. Now condition (iii) gives $\|\mathcal{I}_B\| > 0$ as desired.

5-blocker. By complement symmetry it suffices to consider a $5\,\mathcal{O}$-blocker. Condition (ii) shows there is a white and a black $\mathcal{O}$-core. Since B and W are symmetric we need only show there is a black $\mathcal{I}$-core. The second part of condition (ii) implies $\mathcal{I}[\mathcal{O}_{B'}, \mathcal{I}_W] \neq \emptyset$. With condition (iii) this implies the desired black $\mathcal{I}$-core exists.

6-blocker. We will exhibit four cores that give all the desired relations. The first and third parts of condition (ii) imply condition (ii) of a $5\,\mathcal{O}$-blocker (since $\mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}] \subseteq \mathcal{O}[\mathcal{I}_W, \mathcal{O}_{B'}]$). So we can assume condition (iii) of a $5\,\mathcal{O}$-blocker is not satisfied, i.e., there is a mixed $\mathcal{I}$-core. There is a black $\mathcal{I}$-core, since the third part of condition (ii) is equivalent to $\mathcal{I}[\mathcal{O}_{B'}, \mathcal{I}_{B'}] \neq \emptyset$. Complement symmetry shows there is a black and a mixed $\mathcal{O}$-core. These four cores establish the desired relations. $\square$

We close this section with a useful way of detecting a blocker:

Lemma 4.3 *Suppose $\Phi = \|\mathcal{I}\| = \|\mathcal{O}\|$ and an $\mathcal{O}$-set contains every $\mathcal{I}_{B'}$-core and no $\mathcal{O}_{B'}$-core. Then $\mathcal{F}$ is a 1- or 2-blocker.*

Proof: Let O be the hypothesized $\mathcal{O}$-set. We can assume an $\mathcal{O}$-core N of $\mathcal{O}_{B'}$ is not disjoint from O, else $\mathcal{F}$ is a 2-blocker. The core N does not properly contain the $\mathcal{O}$-set O. The hypothesis implies we must have N and O intersecting and co-disjoint. Hence every $\mathcal{O}$-core other than N is contained in O and so is black. Thus $\|\mathcal{O}_B\| = \Phi - 1$. The $\mathcal{I}$-set $N - O$ contains an $\mathcal{I}$-core, which must be black. Thus $\|\mathcal{I}_B\| = 1$. Now $N \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ shows $\mathcal{F}$ is a 1-blocker. $\square$

5 Incrementation theorem

This section proves our incrementation theorem. Throughout the section we assume $\|\mathcal{I}\|, \|\mathcal{O}\| > 1$, since if not Lemma 2.9 implies the incrementation theorem (Theorem 5.9).

We will convert an arbitrary given family into an equivalent singleton family. We call this process *coloring* the given family. Let I be an $\mathcal{I}$-core. To *make I a singleton* means to choose an element $x \in I$ and make $\{x\}$ an $\mathcal{I}$-set. (If I is a singleton set this operation does nothing.) This operation does not change the cores other than replacing I by x. ($V - x$ is not a new $\mathcal{O}$-core since $\overline{I} \subseteq V - x$.) Clearly a cover of the new family is a cover of the original family. To *make I a black singleton* means to choose a black element $x \in I$ and make I a singleton as above; the core I must belong to $\mathcal{I}_{W'}$ so that x exists.

The following relation is useful for coloring a family. For an arbitrary color B,

$$2\Phi \geq \|\mathcal{I}\| + \|\mathcal{O}\| = \|\mathcal{I}_B\| + \|\mathcal{O}_B\| + \|\mathcal{I}_{B'}\| + \|\mathcal{O}_{B'}\|. \tag{2}$$

Here we use $\|\mathcal{I}\| = \|\mathcal{I}_B\| + \|\mathcal{I}_{B'}\|$. This holds since $\|\mathcal{I}\| > 1$ implies the $\mathcal{I}$-cores are disjoint (Lemma 2.1). Note that the inequality in (2) implies we can say there are at most 2Φ cores (again using the assumption $\|\mathcal{I}\|, \|\mathcal{O}\| > 1$).

It is easy to color a given family without increasing Φ: Make each core into a singleton, choosing at most Φ black (white) elements to be the new cores. This can be done since there are at most 2Φ cores. We get a singleton family with the same value of Φ. The difficulty in coloring is that when the given family is not a blocker the singleton family must not be a 1-blocker.

Lemma 5.1 *Any family has $OPT \leq \Phi + 1$.*

Proof: Make cores into singletons in any way that does not increase Φ. Then create an $\mathcal{I}_B$-core and an $\mathcal{O}_W$-core (recall this operation from Section 3). This increases Φ by 1 and gives a singleton family that is not a 1-blocker. So Lemma 3.5 shows the original family has $OPT \leq \Phi + 1$. $\square$

Lemma 5.2 *If $\Phi > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B and $\|\mathcal{O}\| + \|\mathcal{I}\| < 2\Phi - 1$ then $OPT = \Phi$.*

Proof: The hypotheses imply we can make cores into singletons to get a singleton family $\mathcal{F}'$ where both colors B have $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \leq \Phi - 1$. So $\mathcal{F}'$ is not a 1-blocker and Lemma 3.5 shows $OPT = \Phi$. $\square$

The next lemma shows that in general we can make every black (white) core into a singleton without creating a blocker. We first make a simple observation about condition (*ii*) in blockers. Replace any condition $\mathcal{I}[\mathcal{A}, \mathcal{B}] \neq \emptyset$ in (*ii*) by the equivalent condition $\mathcal{O}[\mathcal{B}, \mathcal{A}] \neq \emptyset$. Then every family in condition (*ii*) of every blocker has one of the forms $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$, $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$, $\mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B]$ or $\mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$, for some colors B, W. Hence every family in condition (*ii*) can be written as

$$\mathcal{O}[\mathcal{I}_B \cup \mathcal{A}, \mathcal{O}_B \cup \mathcal{B}],$$

where $\mathcal{A}$ ($\mathcal{B}$) is either $\emptyset$ or $\mathcal{I}_X$ ($\mathcal{O}_X$). Here B ranges over both colors. As an example condition (*ii*) for a 2-blocker is $\mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}] \neq \emptyset$. By definition $\mathcal{I}_{B'} = \mathcal{I}_W \cup \mathcal{I}_X$ and $\mathcal{O}_{B'} = \mathcal{O}_W \cup \mathcal{O}_X$, as desired.

Lemma 5.3 *Making a black core into a singleton does not give a blocker unless the original family is a blocker.*

Proof: Let $\mathcal{F}$ be the original family, with I an $\mathcal{I}_B$-core containing an element x. Let $\mathcal{F}'$ be the family with I replaced by x as an $\mathcal{I}_B$-core. For the purpose of contradiction assume $\mathcal{F}'$ is a blocker but $\mathcal{F}$ is not. Furthermore by Lemma 4.2 assume $\mathcal{F}'$ is nondegenerate.

$\mathcal{F}$ and $\mathcal{F}'$ have exactly the same cores except for x and I. Hence condition (i) of a blocker for $\mathcal{F}'$ implies (i) for $\mathcal{F}$. The same holds true if the blocker has a condition (iii). We conclude some family in (ii) is nonempty for $\mathcal{F}'$ but empty for $\mathcal{F}$.

First suppose the family has the form $\mathcal{O}[\mathcal{I}_W \cup \mathcal{A},\, \mathcal{O}_W \cup \mathcal{B}]$ (recall the observation before the lemma). Let O be an $\mathcal{O}$-set for $\mathcal{F}'$ contained in the family. Clearly O is not an $\mathcal{O}$-set for $\mathcal{F}$, so $O = V - x$. This implies $\mathcal{O}_W \cup \mathcal{B} = \emptyset$. This makes the blocker degenerate, a contradiction.

Suppose the family has the form $\mathcal{O}[\mathcal{I}_B \cup \mathcal{A},\, \mathcal{O}_B \cup \mathcal{B}]$. Let O be the corresponding $\mathcal{O}$-set of $\mathcal{F}'$. Since $x \in O$, O is an $\mathcal{O}$-set of $\mathcal{F}$. Since $I \not\subseteq O$, Lemma 2.3 implies $O \subset I$. Hence O is black and contains a black $\mathcal{O}$-core. But O is disjoint from every black $\mathcal{O}$-core, a contradiction. $\square$

A family where every nonsingleton core is mixed is called *semi-colored*. To convert any given family into an equivalent semi-colored family, simply make each black (white) core into a singleton. The new family $\mathcal{F}'$ is semi-colored and has the same value of Φ. The lemma shows $\mathcal{F}'$ is not a blocker unless the original family is.

Lemma 5.4 *A family with $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for some color B has $OPT = \Phi$ unless it is a blocker.*

Remark: It is easy to see that if the family is a blocker it is actually a 1- or 2-blocker.

Proof: Assume $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ and the given family is not a blocker. Assume the family $\mathcal{F}$ is semi-colored by the discussion after Lemma 5.3. Make every mixed core into a white singleton. This gives a singleton family $\mathcal{F}'$.

$\Phi(\mathcal{F}') = \Phi(\mathcal{F})$ since (2) implies $\Phi \ge \|\mathcal{I}_{B'}\| + \|\mathcal{O}_{B'}\|$. $\mathcal{F}'$ is obviously not a 1-blocker for B. We will show $\mathcal{F}'$ is not a 1-blocker for W. This completes the argument since then Lemma 3.5 shows $OPT = \Phi$.

Suppose $\mathcal{F}'$ is a 1-blocker for W. Hence $\Phi = \|\mathcal{I}_{B'}\| + \|\mathcal{O}_{B'}\|$. Now the right-hand side of (2) equals 2Φ, which implies $\Phi = \|\mathcal{I}\| = \|\mathcal{O}\|$. Also $\mathcal{F}'$ has $\mathcal{O}[\mathcal{I}_W,\, \mathcal{O}_W] \ne \emptyset$, so take a corresponding $\mathcal{O}$-set O. Since $\mathcal{F}$ is not a 2-blocker, $\mathcal{F}$ has a mixed $\mathcal{I}$-core I, $I \not\subseteq O$, or a mixed $\mathcal{O}$-core N, $N \not\subseteq \overline{O}$. By complement-symmetry assume the former.

O contains the singleton core that belongs to I. Now Lemma 2.3 implies $O \subset I$. Since O contains every $\mathcal{I}_W$-core of $\mathcal{F}'$, $\mathcal{F}$ has exactly one mixed $\mathcal{I}$-core (I) and $\Phi - 1$ black $\mathcal{I}$-cores. Furthermore O contains an $\mathcal{O}$-core of $\mathcal{F}$, which must be black. This must be the only black $\mathcal{O}$-core of $\mathcal{F}$ and $\mathcal{F}$ has $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$. Now $\overline{I}$ makes $\mathcal{F}$ a 1-blocker for B, a contradiction. $\square$

To prove our incrementation theorem we can now assume $\Phi > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B and $\|\mathcal{O}\| + \|\mathcal{I}\| \ge 2\Phi - 1$. Lemmas 5.7 and 5.8 treat the two remaining cases. The overall approach is simple even though the proofs are not short: We will attempt to color the given family $\mathcal{F}$ to get a singleton family $\mathcal{F}'$ that is not a 1-blocker. To do this we define a number of sets. Under certain circumstances each set allows us to construct the desired $\mathcal{F}'$. If none of the sets leads to the desired $\mathcal{F}'$ then we can show that $\mathcal{F}$ is a blocker.

Here are the definitions of the main sets that are used. In general each of these sets need not exist or be unique. B denotes any color and I denotes any mixed $\mathcal{I}$-core.

$$\begin{aligned} X_B &= \mu^*(\mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_B]);\\ I_B &= \mu^*(\mathcal{O}[\mathcal{I}_B \cup \{I\},\, \mathcal{O}_B]);\\ Y_B &= \mu^*(\mathcal{O}[\mathcal{I}_{W'},\, \mathcal{O}_B]). \end{aligned}$$

The fact that $\mathcal{O}$ is crossing gives the following sufficient conditions for these sets to exist: A unique set X_B exists if $\mathcal{I}_B, \mathcal{O}_B, \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$. Suppose this is the case. For any mixed $\mathcal{I}$-core I a unique set I_B exists if $\mathcal{O}[\mathcal{I}_B \cup \{I\}, \mathcal{O}_B] \neq \emptyset$. A unique set Y_B exists if I_B exists for every mixed $\mathcal{I}$-core I. In fact $Y_B = X_B \cup \bigcup\{I_B : I \text{ a mixed } \mathcal{I}\text{-core}\}$.

We define a tuple to be *good* if any of the following conditions holds. In these conditions I (O) ranges over every mixed $\mathcal{I}$-core (every mixed $\mathcal{O}$-core) and B ranges over both colors. Assume a unique set X_B (I_B) exists in the condition involving X_B (I_B).

$$\begin{array}{ll}
(B) & \text{if } \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] = \emptyset; \\
(O,B) & \text{if } \|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 1 \ \text{ and } \ O \subseteq X_B; \\
(I,B) & \text{if } \|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 1,\ \mathcal{I}_B \neq \emptyset \ \text{ and } \ \mathcal{O}[\mathcal{I}_B \cup \{I\}, \mathcal{O}_B] = \emptyset; \\
(I,O,B) & \text{if } \|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2,\ \mathcal{I}_B \neq \emptyset \text{ and } O \subseteq I_B.
\end{array}$$

Two good tuples are *disjoint* if they contain no common entry. In particular one tuple contains B and the other contains W.

Lemma 5.5 *Suppose a semi-colored family $\mathcal{F}$ has $\Phi > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B and $\|\mathcal{O}\| + \|\mathcal{I}\| \ge 2\Phi - 1$.*

(i) If $\mathcal{F}$ has a good tuple containing B then $\mathcal{F}$ has a coloring with $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi$ that is not a 1-blocker for B.

(ii) If $\mathcal{F}$ has two disjoint good tuples then $OPT = \Phi$.

Proof: (i) For each of the four types of tuples we construct a singleton family $\mathcal{F}'$ as follows. If the tuple contains I make it a black singleton. Similarly if the tuple contains O. Then make each remaining mixed core a singleton, choosing colors so that $\mathcal{F}'$ has exactly Φ black cores. This can be done by the definition of good tuple, e.g., if (I, O, B) is good then $\mathcal{F}$ has at most $\Phi - 2$ black cores. $\Phi(\mathcal{F}') = \Phi(\mathcal{F})$ by the hypothesis and (2).

We write $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ assuming the family is $\mathcal{F}$, and $\mathcal{O}'[\mathcal{I}_B, \mathcal{O}_B]$ assuming the family is $\mathcal{F}'$. We now show $\mathcal{F}'$ is not a 1-blocker for B. We consider each of the four types of good tuples in turn. For each type we assume there is a set $N \in \mathcal{O}'[\mathcal{I}_B, \mathcal{O}_B]$ and we derive a contradiction. Clearly $N \in \mathcal{O}'[\mathcal{I}_B, \mathcal{O}_B] \subseteq \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$.

Suppose (B) is good. The assumption $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] = \emptyset$ contradicts the existence of N.

Suppose (O, B) is good. The definition implies $O \subseteq X_B \subseteq N$. Thus N contains the black $\mathcal{O}$-core of $\mathcal{F}'$ contained in O. This contradicts the definition of N.

Suppose (I, B) is good. N contains an element of I (specifically the new black $\mathcal{I}$-core). Since N contains every $\mathcal{I}_B$-core of $\mathcal{F}$ and $\mathcal{I}_B \neq \emptyset$, $N \not\subseteq I$. Now Lemma 2.3 implies $I \subseteq N$. But this contradicts the assumption $\mathcal{O}[\mathcal{I}_B \cup \{I\}, \mathcal{O}_B] = \emptyset$.

Suppose (I, O, B) is good. As in the previous case we get $I \subseteq N$. So the definition of good implies $O \subseteq I_B \subseteq N$. Thus N contains the black $\mathcal{O}$-core of $\mathcal{F}'$ contained in O. This contradicts the definition of N.

(ii) Construct a singleton family $\mathcal{F}'$ by first making the cores in the two good tuples into singletons, as in part (i). Then make each remaining mixed core into a singleton, choosing colors so that $\mathcal{F}'$ has exactly Φ black cores. Part (i) shows $\mathcal{F}'$ is not a 1-blocker for B or W. Hence $OPT = \Phi$ by Lemma 3.5. $\square$

Lemma 5.6 *Suppose a color B has $\mathcal{I}_B, \mathcal{O}_B \neq \emptyset$ and $\Phi > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$. Either some good tuple contains B or these properties all hold:*

(i) A unique set X_B exists and equals $\mu^*(\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}])$.

(ii) Every mixed $\mathcal{I}$-core I has a unique set I_B.
(iii) If $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2$ then $Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$.

Proof: Assume $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$ else (B) is good. Now a unique set X_B exists. If X_B contains a mixed $\mathcal{O}$-core O then (O, B) is good. In the remaining case no mixed $\mathcal{O}$-core O is contained in X_B, and Lemma 2.5 shows O and X_B are disjoint. Hence $X_B \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$, giving (i).

If $\mathcal{O}[\mathcal{I}_B \cup \{I\}, \mathcal{O}_B] = \emptyset$ for a mixed $\mathcal{I}$-core I then (I, B) is good. In the opposite case a unique I_B exists. Part (ii) follows.

Suppose $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2$. If some mixed $\mathcal{O}$-core is contained in some set I_B then (I, O, B) is good. If not, every set I_B is disjoint from every mixed $\mathcal{O}$-core, by Lemma 2.5. Thus $Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$, giving part (iii). □

Lemma 5.7 *A family with $\Phi = \|\mathcal{O}\| > \|\mathcal{I}\|$, $\|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B has $OPT = \Phi$ unless it is a 5- or 6-blocker.*

Proof: The hypotheses show we can assume the given family is not a blocker of any type. Assume the family $\mathcal{F}$ is semi-colored, by the discussion after Lemma 5.3. By Lemma 5.2 assume there are exactly $2\Phi - 1$ cores. Hence any coloring has exactly Φ cores of one color and $\Phi - 1$ cores of the other. So it suffices to find a good tuple, since then Lemmas 5.5 and 3.5 show $OPT = \Phi$.

Assume some color B has $\mathcal{I}_B, \mathcal{O}_B \neq \emptyset$ by Corollary 2.11. Lemma 5.6(i)–(ii) shows we can assume there are unique sets

$$X_B \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}],\ Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B].$$

X_B contains an $\mathcal{O}$-core so $\mathcal{O}_W \neq \emptyset$. $V - Y_B$ contains an $\mathcal{I}$-core so $\mathcal{I}_W \neq \emptyset$. Now as in the previous paragraph we get

$$X_W \in \mathcal{O}[\mathcal{I}_W, \mathcal{O}_{B'}],\ Y_W \in \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_W].$$

If $\|\mathcal{I}_X\| = 0$ then X_B and X_W show $\mathcal{F}$ is a 5 $\mathcal{O}$-blocker. If $\|\mathcal{O}_X\| = 0$ then $V - Y_B \in \mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$ and $V - Y_W \in \mathcal{I}[\mathcal{O}_W, \mathcal{I}_{B'}]$ show $\mathcal{F}$ is a 5 $\mathcal{I}$-blocker. Hence assume $\|\mathcal{I}_X\|, \|\mathcal{O}_X\| \ge 1$.

The assumption implies Y_B and Y_W have nonempty intersection. Hence Lemma 2.4 shows they are co-disjoint. Lemma 2.5 shows one of them, say Y_B, contains a mixed $\mathcal{O}$-core O. Thus some set I_B contains O. Also Lemma 5.6(iii) shows $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1$.

The latter implies $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| \le \Phi - 2$. So Lemma 5.6$(iii)$ shows $Y_W \in \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$. Now $X_B \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$, $V - Y_B \in \mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$ and $Y_W \in \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$ make $\mathcal{F}$ a 6-blocker for B. □

Lemma 5.8 *A family with $\Phi = \|\mathcal{I}\| = \|\mathcal{O}\| > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B has $OPT = \Phi$ unless it is a 2-, 3- or 4-blocker.*

Proof: The hypotheses show we can assume the given family is not a blocker. Assume the family $\mathcal{F}$ is semi-colored, by the discussion after Lemma 5.3. There are exactly 2Φ cores. Hence any coloring has Φ cores of each color. If we construct such a coloring that is not a 1-blocker for B or W then Lemma 3.5 shows $OPT = \Phi$.

Assume B has at least one $\mathcal{I}$-core and at least one $\mathcal{O}$-core, by Corollary 2.11. Not every $\mathcal{I}$-core is black, nor every $\mathcal{O}$-core, by hypothesis.

Case 1. $\mathcal{O}_W = \emptyset$.

We will define a coloring $\mathcal{F}'$ that shows $OPT = \Phi$ or else discover a 1- or 2-blocker. Throughout this discussion "core" refers to $\mathcal{F}$ and "singleton" refers to $\mathcal{F}'$. We first specify $\mathcal{F}'$ and then show it has the desired properties. $\mathcal{F}'$ is constructed in three steps.

Step 1. Make every mixed $\mathcal{I}$-core a white singleton.

After Step 1 an $\mathcal{O}$-set O contains every $\mathcal{I}_{B'}$-core if and only if O contains every white singleton. In proof first observe $\|\mathcal{I}_{B'}\| > 1$. (Otherwise there are at least $\Phi - 1$ black $\mathcal{I}$-cores and a black $\mathcal{O}$-core, contradicting the lemma's hypothesis.) If O contains every white singleton but not every $\mathcal{I}_{B'}$-core then Lemma 2.3 implies $O \subset I$ for some mixed $\mathcal{I}$-core I. So I contains an element of another $\mathcal{I}$-core, a contradiction.

We can now make up to $\|\mathcal{I}_B\|$ mixed $\mathcal{O}$-cores into white singletons, since $\Phi = \|\mathcal{I}\| = \|\mathcal{I}_B\| + \|\mathcal{I}_{B'}\|$. We do this by making at most one mixed $\mathcal{O}$-core white for each black $\mathcal{I}$-core, as follows.

Step 2. For each black $\mathcal{I}$-core I do the following. Define

$$Z_I = \mu^*(\mathcal{O}[\mathcal{I}_{B'}, \{I\}]).$$

Z_I is uniquely defined since $\mathcal{O}$ is crossing, $\mu(\mathcal{I}_{B'}) \neq \emptyset$ and $V - I \in \mathcal{O}[\mathcal{I}_{B'}, \{I\}]$. Z_I contains a mixed $\mathcal{O}$-core, since otherwise it contains no $\mathcal{O}_{B'}$-core (by Case 1) and Lemma 4.3 shows $\mathcal{F}$ is a 1- or 2-blocker. Choose a mixed $\mathcal{O}$-core $N \subseteq Z_I$, where if possible $N \not\subseteq X_B$ when the set X_B exists. Make N a white singleton.

Step 3. Make the remaining cores into singletons so both B and W get exactly Φ singletons, and if possible some $\mathcal{O}$-core contained in X_B becomes a black singleton.

Now we show $\mathcal{F}'$ is not a 1-blocker. Suppose $\mathcal{F}'$ is a 1-blocker for W. Hence some $\mathcal{O}$-set O contains every white $\mathcal{I}$-singleton but no white $\mathcal{O}$-singleton. Let I be an $\mathcal{I}$-core contained in $V - O$. The remark after Step 1 shows $O \in \mathcal{O}[\mathcal{I}_{B'}, \{I\}]$. Hence I is black and $Z_I \subseteq O$. But Z_I contains a white $\mathcal{O}$-singleton, a contradiction.

Next suppose $\mathcal{F}'$ is a 1-blocker for B. Hence some $\mathcal{O}$-set O contains every black $\mathcal{I}$-singleton but no black $\mathcal{O}$-singleton. This implies the set X_B exists and $X_B \subseteq O$. So we need only show that X_B contains a black $\mathcal{O}$-singleton to get the desired contradiction.

Every $\mathcal{O}$-core $N \subseteq X_B$ is mixed (Case 1). Take such an N and suppose that Step 2, when executed for the black $\mathcal{I}$-core I, made N a white singleton. Thus $N \subseteq X_B \cap Z_I$. $X_B \cup Z_I$ contains every $\mathcal{I}$-core, so Lemma 2.4 shows X_B and Z_I are co-disjoint.

Every mixed $\mathcal{O}$-core contained in Z_I is contained in X_B by Step 2. Now co-disjointness and Lemma 2.5 show X_B contains every mixed $\mathcal{O}$-core. The hypothesis $\Phi = \|\mathcal{O}\| > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ and Case 1 imply $\|\mathcal{O}_X\| > \|\mathcal{I}_B\|$. Hence Step 2 does not make every mixed $\mathcal{O}$-core contained in X_B a white singleton. Thus Step 3 makes some $\mathcal{O}$-core contained in X_B a black singleton, as desired.

This completes the analysis of Case 1. The case $\mathcal{I}_W = \emptyset$ follows by symmetry. Hence from now on assume

$$\mathcal{I}_B, \mathcal{O}_B, \mathcal{I}_W, \mathcal{O}_W \neq \emptyset.$$

If $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] = \mathcal{O}[\mathcal{I}_W, \mathcal{O}_W] = \emptyset$ then (B) and (W) are good and Lemma 5.5 shows $OPT = \Phi$.

Case 2. $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$, $\mathcal{O}[\mathcal{I}_W, \mathcal{O}_W] = \emptyset$.

Since (W) is good we need only find a good tuple containing B (Lemma 5.5(ii)). Suppose there is no such tuple. Lemma 5.6(i)–(ii) give the sets $X_B = \mu^*(\mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}])$ and I_B for every mixed $\mathcal{I}$-core I. Hence we also have the set $Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B]$.

Suppose $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1$. The sets $X_B \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$ and $V - Y_B \in \mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$ make $\mathcal{F}$ a 4-blocker. So $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2$. Lemma 5.6$(iii)$ shows $Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$. This makes $\mathcal{F}$ a 2-blocker.

This completes Case 2. Hence we can assume the remaining possibility,

$$\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B],\ \mathcal{O}[\mathcal{I}_W, \mathcal{O}_W] \neq \emptyset.$$

Observe that sets X_B and X_W exist.

Case 3. $\|\mathcal{O}_X\| = 0$.

If Y_B exists then $Y_B \in \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_B] = \mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$. This makes $\mathcal{F}$ a 2-blocker. We conclude that there are $\mathcal{I}$-cores I, J such that neither of I_B, J_W exists, i.e., (I, B) and (J, W) are good. If I and J can be chosen to be distinct Lemma 5.5 shows $OPT = \Phi$.

We will show such I and J always exist. Suppose not. Case 3 implies $\|\mathcal{I}_X\| \ge 2$. Hence we can assume I and J are $\mathcal{I}$-cores where I_B and I_W exist but neither J_B nor J_W exists. $I_B \cap I_W$ contains I but no $\mathcal{O}$-core, since I_B (I_W) contains only white (black) $\mathcal{O}$-cores. Lemma 2.4 shows I_B and I_W are intersecting and co-disjoint. Lemma 2.3 shows J is not intersecting with I_B or I_W. Hence J is contained in one of these sets, say I_B. But this implies J_B exists, a contradiction.

This completes Case 3. By symmetry we can now assume

$$\|\mathcal{O}_X\|, \|\mathcal{I}_X\| > 0.$$

Claim 1: If $\mathcal{F}$ is not a 2-blocker then any color B with at most $\Phi - 2$ cores has a good tuple (I, B) or (I, O, B) (for I (O) a mixed $\mathcal{I}$-core (mixed $\mathcal{O}$-core)).

Proof: Suppose the claim fails. So for every mixed $\mathcal{I}$-core I, I_B exists and is disjoint from every mixed $\mathcal{O}$-core (Lemma 2.5). Thus Y_B exists and belongs to $\mathcal{O}[\mathcal{I}_{W'}, \mathcal{O}_{W'}]$. This makes $\mathcal{F}$ a 2-blocker. ♠

Claim 2: If (O, B) is good for a mixed $\mathcal{O}$-core O, $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| \le \Phi - 2$ and $\|\mathcal{O}_X\| \ge 2$ then $OPT = \Phi$.

Proof: Lemma 5.5 shows we can assume that no mixed $\mathcal{I}$-core I has (I, W) good. Hence I_W exists for every I and also Y_W exists. Furthermore Claim 1 shows I_W contains a mixed $\mathcal{O}$-core for some mixed $\mathcal{I}$-core I. If we can choose the mixed $\mathcal{O}$-cores $O \subseteq X_B$ and $N \subseteq I_W$ to be distinct then Lemma 5.5 shows $OPT = \Phi$.

To show such distinct sets N and O always exist suppose the contrary. Note that I ranges over every mixed $\mathcal{I}$-core. So the supposition implies there is a unique mixed $\mathcal{O}$-core O contained in $X_B \cap Y_W$, and no other mixed $\mathcal{O}$-core is contained in $X_B \cup Y_W$. (Use Lemma 2.5.) We have $O \subseteq X_B \cap Y_W$ and $X_B \cup Y_W$ contains every $\mathcal{I}$-core. Lemma 2.4 shows X_B and Y_W are co-disjoint. This makes O the only mixed $\mathcal{O}$-core, contradicting $\|\mathcal{O}_X\| \ge 2$. ♠

Claim 3: If (I, B) is good for a mixed $\mathcal{I}$-core I, $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| \le \Phi - 2$ and $\|\mathcal{I}_X\| \ge 2$ then $OPT = \Phi$.

Proof: This is the complement-symmetric version of Claim 2: Let Z_B be the maximal set of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$. (I, B) good implies $I \not\subseteq Z_B$. Lemma 2.3 easily implies that I is disjoint from Z_B. Equivalently $I \subseteq \mu^*(\mathcal{I}[\mathcal{O}_B, \mathcal{I}_B])$ and (I, B) is good in the complementary family. Claim 2 applied to that family shows $OPT = \Phi$. ♠

Case 4. $\|\mathcal{I}_X\| = 1$.

Let I be the unique mixed $\mathcal{I}$-core. Suppose

$$X_W \text{ contains a mixed } \mathcal{O}\text{-core } O.$$

Thus (O, W) is good. If I_B does not exist then (I, B) is good so Lemma 5.5(ii) shows $OPT = \Phi$. Hence we can assume

$$I_B \text{ exists.}$$

If $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2$ then Case 4 implies $\|\mathcal{O}_X\| \ge 2$. Now Claim 2 shows $OPT = \Phi$. Hence we can assume

$$\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1.$$

If X_B contains no mixed $\mathcal{O}$-core then $X_B \in \mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}]$ by Lemma 2.5. Also $V - I_B \in \mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}]$. This makes $\mathcal{F}$ a 4-blocker. Hence we can assume X_B contains a mixed $\mathcal{O}$-core.

Now we can interchange B and W in the previous paragraph. Thus we can assume that I_W exists and $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| = \Phi - 1$. This makes $\mathcal{F}$ a $3\,\mathcal{I}$-blocker since $V - I_B \in \mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}]$, $V - I_W \in \mathcal{I}[\mathcal{O}_W,\, \mathcal{I}_{B'}]$, and Case 4 implies $\|\mathcal{O}_X\| = 1$.

The remaining possibility is that neither X_B nor X_W contains a mixed $\mathcal{O}$-core. X_B is disjoint from every mixed $\mathcal{O}$-core by Lemma 2.5. Hence $X_B \in \mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}]$. Similarly $X_W \in \mathcal{O}[\mathcal{I}_W,\, \mathcal{O}_{B'}]$. This with Case 4 makes $\mathcal{F}$ a $3\,\mathcal{O}$-blocker.

This completes Case 4. Using symmetry we can assume

$$\|\mathcal{O}_X\|, \|\mathcal{I}_X\| \ge 2.$$

Case 5. $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| = \Phi - 1$.

Since there are at least 4 mixed cores we get $\|\mathcal{I}_W\| + \|\mathcal{O}_W\| < \Phi - 2$. If X_B contains a mixed $\mathcal{O}$-core then Claim 2 shows $OPT = \Phi$. If I_B does not exist for some mixed $\mathcal{I}$-core I then Claim 3 shows $OPT = \Phi$. Now we can assume X_B contains no mixed $\mathcal{O}$-core and Y_B exists. Lemma 2.5 shows $X_B \in \mathcal{O}[\mathcal{I}_B,\, \mathcal{O}_{W'}]$, and $V - Y_B \in \mathcal{I}[\mathcal{O}_B,\, \mathcal{I}_{W'}]$. This makes $\mathcal{F}$ a 4-blocker.

This completes Case 5. By symmetry the one remaining case is this:

Case 6. Both colors B satisfy $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \le \Phi - 2$.

If I_B does not exist for some $\mathcal{I}$-core I then Claim 3 applies. Hence I_B and I_W exist for every $\mathcal{I}$-core I. Also Y_B and Y_W exist.

We first show any mixed $\mathcal{I}$-core I is in at least two good triplets. For any mixed $\mathcal{I}$-core I either

(a) $I_B \cap I_W$ contains a mixed $\mathcal{O}$-core, or
(b) I_B and I_W are intersecting and co-disjoint.

In proof suppose (a) fails. Then $I_B \cap I_W$ contains no $\mathcal{O}$-core (since every black (white) $\mathcal{O}$-core is disjoint from I_B (I_W)). Lemma 2.4 gives (b).

In case (a) if O is a common mixed $\mathcal{O}$-core then (I, O, B) and (I, O, W) are good. In case (b) any mixed $\mathcal{O}$-core is contained in I_B or I_W (Lemma 2.5). I is in at least two good triplets since we have assumed $\|\mathcal{O}_X\| \ge 2$.

We will now show there are two disjoint good triplets, i.e., (I, O, B) and (J, N, W) with $I \ne J$ and $O \ne N$. This will complete the proof by Lemma 5.5. We consider three cases.

Case 6.1. For some mixed $\mathcal{I}$-core I, $I_B - I_W$ contains every mixed $\mathcal{O}$-core O.

I does not belong to any good triplet $(I, \cdot, W)$. Claim 1 applied to W implies some good triplet contains W, say (J, N, W). Clearly $J \ne I$. We can choose a good triplet (I, O, B) with $O \ne N$ since I belongs to two good triplets.

Case 6.2. For some mixed $\mathcal{I}$-core I satisfying (b) both $I_B - I_W$ and $I_W - I_B$ contain a mixed $\mathcal{O}$-core.

Take any mixed $\mathcal{I}$-core $J \neq I$. J is contained in at least one of I_B, I_W (Lemma 2.3), say $J \subseteq I_B$. This implies $J_B \subseteq I_B$. Suppose (J, N, B) is good for some $\mathcal{O}$-core N. Let O be an $\mathcal{O}$-core in $I_W - I_B$. Clearly $O \neq N$ and (I, O, W) is good.

In the remaining case any good triplet containing J contains W. Let O be an $\mathcal{O}$-core in $I_B - I_W$. So (I, O, B) is good. We can choose a good triplet (J, N, W) with $N \neq O$ since J is in two good triplets.

If no mixed $\mathcal{I}$-core satisfies the condition of Case 6.1 or 6.2 then every mixed $\mathcal{I}$-core satisfying (b) also satisfies (a) (Lemma 2.5). Hence there is one remaining case:

Case 6.3. Every mixed $\mathcal{I}$-core satisfies (a).

Consider the set of pairs

$$P = \{(I, O) : \text{ both } (I, O, B) \text{ and } (I, O, W) \text{ are good triplets}\}.$$

Every mixed $\mathcal{I}$-core I belongs to a pair of P, by Case 6.3. If two distinct $\mathcal{O}$-cores belong to pairs of P then we are done. (In proof suppose for $O \neq O'$ there are good triplets (I, O, B), (I, O, W), (I', O', B), (I', O', W). If $I \neq I'$ we clearly have two disjoint good triplets. If $I = I'$ then for any mixed $\mathcal{I}$-core $J \neq I$, any good triplet (J, N, W) is disjoint from one of these four triplets. J exists since $\|\mathcal{I}_X\| \geq 2$.) Hence assume every pair of P contains the same mixed $\mathcal{O}$-core O.

The sets Y_B and Y_W exist and are co-disjoint (Lemma 2.4). Hence every mixed $\mathcal{O}$-core is in Y_B or Y_W (Lemma 2.5). Without loss of generality Y_W contains a mixed $\mathcal{O}$-core $N \neq O$. Hence there is a good triplet (J, N, W) for some mixed $\mathcal{I}$-core J. Choose $I \neq J$ to get the desired good triplet (I, O, B). □

Theorem 5.9 *Any family has $OPT = \Phi$ unless it is a blocker, in which case $OPT = \Phi + 1$.*

Proof: A blocker has $OPT = \Phi + 1$ by Lemmas 4.1 and 5.1. If $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for some color B the theorem follows from Lemma 5.4. So assume $\Phi > \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B. If there are fewer than $2\Phi - 1$ cores then $OPT = \Phi$ by Lemma 5.2. So assume there are $2\Phi - 1$ or 2Φ cores. These two cases are handled by Lemmas 5.7 and 5.8 respectively. □

6 Related results

6.1 Bipartite strong connectivity incrementation

Reference [8] solves the special case of our problem where we wish to make a digraph strongly connected by adding edges that respect a given bipartition. The min-max theorem of [8] is analogous to ours except that there are no 5- or 6-blockers. We show this is true in a more general sense. Suppose $\mathcal{I}$ is a ring family. $\mathcal{O}$ is also a ring family, and we will call $\mathcal{F}$ a ring family. Both $\emptyset$ and V can belong to $\mathcal{I}$ or $\mathcal{O}$. To consider our covering problem on $\mathcal{F}$ we remove these two sets from both $\mathcal{I}$ and $\mathcal{O}$. We show such families do not need 5- or 6-blockers:

Lemma 6.1 *If $\mathcal{F}$ is a ring family then a 5- or 6-blocker is also a 1-blocker.*

Proof:

5-blocker. Consider a $5\,\mathcal{O}$-blocker. Take sets $O \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$, $N \in \mathcal{O}[\mathcal{I}_W, \mathcal{O}_{B'}]$. If $\overline{O} \cap \overline{N}$ is nonempty then it is an $\mathcal{I}$-set since $\mathcal{I}$ is a ring family. But it contains no $\mathcal{I}$-core since $\|\mathcal{I}_X\| = 0$. Hence $O \cup N = V$. Lemma 2.5 shows every $\mathcal{O}$-core is black or white, and no mixed core exists. Hence one of the colors, say B, has $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$. Since $O \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ this makes $\mathcal{F}$ a 1-blocker.

The case of a $5\,\mathcal{I}$-blocker follows by complement symmetry, since a family is a 1-blocker if and only if its complement is.

6-blocker. Take sets $O \in \mathcal{O}[\mathcal{I}_B, \mathcal{O}_{W'}]$, $I \in \mathcal{I}[\mathcal{O}_B, \mathcal{I}_{W'}]$, $N \in \mathcal{O}[\mathcal{I}_{B'}, \mathcal{O}_{B'}]$. We first show no mixed core exists. If $\overline{O} \cap \overline{N}$ is nonempty then it is an $\mathcal{I}$-set since $\mathcal{I}$ is a ring family. But inspection shows it contains no $\mathcal{I}$-core. Hence $O \cup N = V$. As in the previous case no mixed $\mathcal{O}$-core exists. Similarly if $N \cap \overline{I}$ is nonempty then it is an $\mathcal{O}$-set. But inspection shows it contains no $\mathcal{O}$-core. Hence $N \subseteq I$ and no mixed $\mathcal{I}$-core exists.

This implies that $\Phi = \|\mathcal{I}_W\| + \|\mathcal{O}_W\|$. Since $N \in \mathcal{O}[\mathcal{I}_W, \mathcal{O}_W]$ $\mathcal{F}$ is a 1-blocker. $\square$

6.2 Undirected bipartite incrementation

As already mentioned [3] solves the undirected version of our problem, i.e., the problem of incrementing the edge-connectivity of an undirected graph using undirected bipartite edges. To compare the results suppose we wish to increase the connectivity from k to $k+1$. Let $\mathcal{I}$ be the family of vertex sets of degree exactly k. Define

$$\Phi_u = \max\{\lceil \|\mathcal{I}\|/2 \rceil, \|\mathcal{I}_B\| : B \in \mathcal{P}\}.$$

[3] shows that for $k \geq 1$, $OPT = \Phi_u$ unless the graph is a so-called C_4-configuration, in which case $OPT = \Phi_u + 1$. A C_4-configuration has a number of defining properties. We give only three of the properties, for the current context of connectivity incrementation:

(i) k is even (i.e., the target connectivity $k+1$ is odd).
(ii) There are 4 $\mathcal{I}$-zones that form a partition of V.
(iii) $\Phi_u = \|\mathcal{I}\|/2 = \|\mathcal{I}_B\| = 2$ for some color B.

These properties illustrate the different flavor of our results from [3]: Our blockers exist for every value of k. Furthermore Φ, as well as the number of zones, can be arbitrarily large in a blocker.

We can make another comparison by viewing undirected graphs as digraphs. We consider an undirected edge to be two oppositely directed edges. The family $\mathcal{I}$ in the definition of Φ_u does not change if we interpret it as the family of sets of in-degree k in the given graph. The result of [3] gives a cover that consists of either $2\Phi_u$ or $2\Phi_u + 2$ directed edges.

To apply our results to the same problem consider a crossing family that is self-complementary, i.e., $\mathcal{I} = \mathcal{O}$. Such a family arises in the connectivity incrementation problem for an Eulerian digraph (i.e., a digraph where every vertex has its in-degree equal to its out-degree). A special case is an undirected graph viewed as a digraph, as above. A self-complementary family has $\Phi = \max\{\|\mathcal{I}\|, 2\|\mathcal{I}_B\| : B \in \mathcal{P}\}$. It is easy to check that such a family has no blocker (use the fact that a blocker is nondegenerate). Hence $OPT = \Phi$. Now comparing Φ with $2\Phi_u$ we see that the undirected solution of [3], considered as a directed solution, uses at most 1 more edge than the optimum directed solution unless the graph is a C_4-configuration, which uses 2 more edges.

6.3 Generalized theorem

We generalize our covering problem by allowing "mixed" elements (that can be joined to other mixed or colored elements) and "unavailable" elements (that cannot be joined to any other elements). More formally consider a crossing family $\mathcal{I}$ on a set of elements V, along with a partition $\mathcal{P}$ of V into four blocks, B, W, M, U. The *generalized bipartitioned crossing-family covering problem* is to cover every set of $\mathcal{I}$ using the smallest number of directed edges, subject to the restriction that an edge is not incident to an element of U and does not join two elements of B or of W. Hence an edge joins two blocks of B, W, M or joins two elements in M.

We extend our results to this case. Redefine $\mathcal{I}_B$ to be the family of all $\mathcal{I}$-sets contained in $B \cup U$. Call a set of elements *mixed* if it contains an element of M or elements of both B and W. $\mathcal{I}_X$ is now the family of $\mathcal{I}$-sets that do not contain a core of $\mathcal{I}_B$ or of $\mathcal{I}_W$. All other definitions (e.g., blockers) are unchanged.

Corollary 6.2 *The generalized bipartitioned crossing-family covering problem has a solution as long as $\emptyset, V \notin \mathcal{I}$; $M \neq \emptyset$ or $B, W \neq \emptyset$; and no core is contained in U. In this case $OPT = \Phi$ unless the family is a blocker, in which case $OPT = \Phi + 1$.*

Proof: Model the problem as an instance of our original covering problem, using the crossing family operations of Section 2.4 as follows. Duplicate each element of M, making one copy black and the other copy white. Delete every element of U. $\square$

Frank [5] solved the (unrestricted) edge-connectivity augmentation problem for digraphs. Specialized to connectivity incrementation his result is that $OPT = \max\{\|\mathcal{I}\|, \|\mathcal{O}\|\}$. The corollary contains this result as a special case. (Take every element to be mixed. It is easy to see there is no blocker in this case, since $\mathcal{O}_{B'} = \mathcal{O}_{W'} = \mathcal{O}$.)

6.4 Other crossing families

We illustrate our results on two other crossing families. The following fact is useful to verify the examples.

Lemma 6.3 *Consider a feasible generalized bipartitioned crossing-family covering problem.*

(i) A blocker with no mixed core is a 1-blocker.

(ii) A singleton $\mathcal{I}$-set or $\mathcal{O}$-set does not belong to any family $\mathcal{O}[\cdot, \cdot]$ or $\mathcal{I}[\cdot, \cdot]$ in condition (ii) of a nondegenerate blocker.

(iii) In a nondegenerate $3\,\mathcal{O}$-, $5\,\mathcal{O}$- or 6-blocker no $\mathcal{O}$-core is contained in every nonsingleton $\mathcal{O}$-set. In fact an $\mathcal{O}$-core is contained in $\mathcal{O}$-sets that belong to at most one of the two families $\mathcal{O}[\cdot, \cdot]$ in condition (ii).

Proof: (i) For 2- and 4-blockers use $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for both colors B. For 5- and 6-blockers use $\Phi = \|\mathcal{I}_B\| + \|\mathcal{O}_B\|$ for one color B.

(ii) The family assigns contradictory colors to the singleton. To illustrate consider a 1-blocker. A singleton in $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ must be black since it is a core of $\mathcal{I}_B$. But that makes it a black $\mathcal{O}$-core, a contradiction.

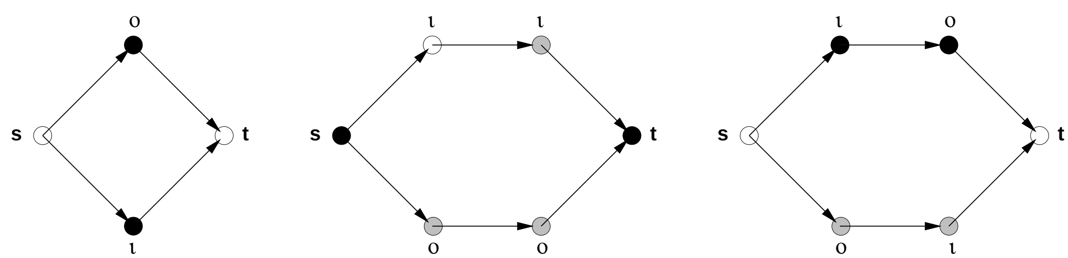
(iii) For each of the blocker types listed, condition (ii) has two families of the form $\mathcal{O}[\cdot, \cdot]$. In fact the two families have the form $\mathcal{O}[\cdot, \mathcal{O}_{W'}]$ and $\mathcal{O}[\cdot, \mathcal{O}_{B'}]$. Let O be an $\mathcal{O}$-core contained in sets of both families. The first family implies $O \in \mathcal{O}_W$ and the second implies $O \in \mathcal{O}_B$. This makes $O \subseteq U$, contradicting feasiblity. $\square$

We first look at a flow problem. Consider a flow network with source s, sink t and an integral capacity function. The vertex set V is given with a bipartition, along with two sets $S_\iota, S_o \subseteq V$. We wish to add the smallest number of new edges, each respecting the bipartition, so the value of the maximum flow from s to t increases and each vertex in S_ι (S_o) has a new edge directed into it (out of it).

Let $\mathcal{O}$ be the family of all s,t-mincuts, i.e., those sets containing s but not t whose out-degree equals the maximum flow value. $\mathcal{O}$ is a crossing family, with complementary family $\mathcal{I}$. Make each element of S_ι (S_o) a singleton $\mathcal{I}$-set ($\mathcal{O}$-set). We get a bipartitioned crossing-family covering problem that is equivalent to the flow problem.

We define two natural families containing all the $\mathcal{I}$-cores and all the $\mathcal{O}$-cores respectively. Let σ be the minimal s,t-mincut. Define the family $\mathcal{O}^*$ to consist of σ and $\{v\}$ for every $v \in S_o$. Similarly let τ be the complement of the maximal s,t-mincut and define $\mathcal{I}^*$ to consist of τ and $\{v\}$ for every $v \in S_\iota$. Our lower bound becomes $\Phi = \max\{\|\mathcal{I}^*\|, \|\mathcal{O}^*\|, \|\mathcal{I}^*_B\| + \|\mathcal{O}^*_B\| : B \in \mathcal{P}\}$. Our theorem says that $OPT = \Phi$ unless the network is a blocker, in which case $OPT = \Phi + 1$.

Lemma 6.3(ii)–(iii) implies this problem has no nondegenerate 3-, 5- or 6-blocker (since σ (τ) is contained in every nonsingleton $\mathcal{O}$-set ($\mathcal{I}$-set)). Assuming there are no mixed vertices we can eliminate 2- and 4-blockers using Lemma 6.3(i) (σ (τ) cannot be a mixed $\mathcal{O}$-core (mixed $\mathcal{I}$-core) in a 2- or 4-blocker). Hence a 1-blocker is the only nondegenerate blocker that can exist. Fig. 3(a) shows such a blocker. If we allow mixed vertices then there can be 2- and 4-blockers, as illustrated in Fig. 3(b)–(c).

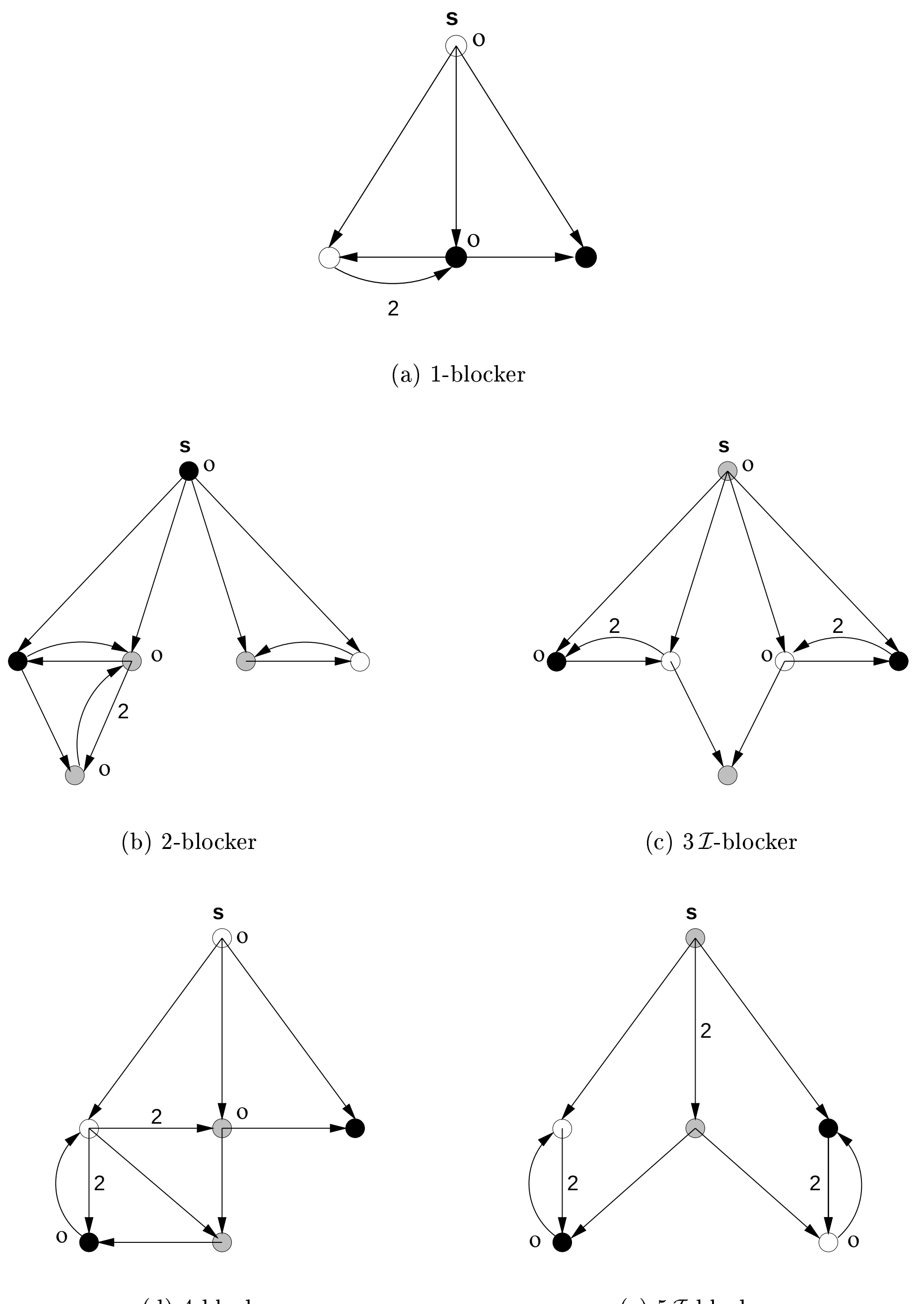


(a) 1-blocker (b) 2-blocker (c) 4-blocker

Figure 3: Blockers for the flow problem. Each edge has capacity 1 and the maximum flow value is 2. Vertices in S_ι (S_o) are labelled by ι (o). Every vertex is a singleton $\mathcal{I}$- or $\mathcal{O}$-core (but not both) and there are no other cores. (a) has $\Phi = 2$. (b)–(c) have $\Phi = 3$ and satisfy only one set of blocker conditions.

We turn to a problem on arborescences. Consider a digraph with a distinguished vertex s. An *s-arborescence* is a spanning tree directed outwards from its root s. As before the vertex set V is given with a bipartition and sets $S_\iota, S_o \subseteq V$. We wish to add the smallest number of new edges, each respecting the bipartition, so the number of edge-disjoint s-arborescences increases and each vertex in S_ι (S_o) has a new edge directed into it (out of it).

Let k be the greatest number of edge-disjoint s-arborescences. Let $\mathcal{I}$ be the family of all sets disjoint from s with in-degree exactly k. $\mathcal{I}$ is a crossing family. Let $\mathcal{O}$ be the complementary family. Make each element of S_ι (S_o) a singleton $\mathcal{I}$-set ($\mathcal{O}$-set). This gives a bipartitioned crossing-family covering problem equivalent to the arborescence problem by Edmonds' Theorem [4]. Our theorem says that $OPT = \Phi$ unless the graph is a blocker, in which case $OPT = \Phi + 1$. Fig. 4 illustrates all the blocker types except $3\,\mathcal{O}$-, $5\,\mathcal{O}$- and 6-blockers. We now show these last three types do not



(a) 1-blocker

(b) 2-blocker

(c) 3 $\mathcal{I}$-blocker

(d) 4-blocker

(e) 5 $\mathcal{I}$-blocker

Figure 4: Blockers for the arborescence problem. Edges with multiplicity > 1 are labelled. Each graph has 2 disjoint s-arborescences. Vertices of S_ι (S_o) are labelled by ι (o). Except for s in (e), every vertex is a singleton $\mathcal{I}$- or $\mathcal{O}$-core but not both, and there are no other cores. (a) has $\Phi = 2$ and (b)–(e) have $\Phi = 3$. (b)–(e) satisfy only one set of the 5 possible blocker conditions.

exist for the arborescence problem.

More precisely we will show that any nondegenerate $3\,\mathcal{O}$-, $5\,\mathcal{O}$- or 6-blocker is also a blocker of a different type. To do this it suffices to prove $\|\mathcal{O}_X\| = 0$. This relation implies that for a $3\,\mathcal{O}$- or $5\,\mathcal{O}$-blocker some color satisfies condition (i) of a 1-blocker, and condition (ii) of the original blocker implies (ii) of a 1-blocker. For a 6-blocker it is easy to see condition (ii) implies condition (ii) of a $5\,\mathcal{I}$-blocker, and condition (iii) of a $5\,\mathcal{I}$-blocker is our relation.

Now we prove $\|\mathcal{O}_X\| = 0$. Lemma 6.3(ii)-(iii) shows there are two distinct nonsingleton $\mathcal{O}$-sets in the two families $\mathcal{O}[\cdot, \mathcal{O}_{W'}]$ and $\mathcal{O}[\cdot, \mathcal{O}_{B'}]$ of condition (ii). Any mixed $\mathcal{O}$-core is clearly disjoint from both $\mathcal{O}$-sets. Also both $\mathcal{O}$-sets contain s. Hence the intersection of the two $\mathcal{O}$-sets is an $\mathcal{O}$-set. So the intersection contains an $\mathcal{O}$-core, violating Lemma 6.3(iii).

7 Efficient algorithm

We show the bipartite digraph edge-connectivity incrementation problem can be solved in the same time as the best-known algorithm for unrestricted digraph edge-connectivity incrementation: $O(km \log n)$ for unweighted digraphs and $O(nm \log (n^2/m))$ for weighted digraphs, for k the target connectivity [7]. These bounds hold even when the starting graph does not respect the given bipartition.

The algorithm follows our proof of the incrementation theorem. It begins by coloring the graph, following Section 5. It solves the covering problem on the resulting singleton family following Section 3. This in turn uses routines based on Section 2.

Each part of the algorithm follows the proof of the corresponding lemma. We will describe the basic tools used to make the algorithm efficient. We will also give implementations of all the proofs where achieving the desired time bound is not immediate. The proofs that are implemented in a straightforward way are not discussed. We present the algorithms in the same order as the lemmas: Subsections 7.1–7.5 correspond to Section 2, Subsections 7.6–7.7 correspond to Section 3, Subsection 7.8 corresponds to Section 4 and Subsections 7.9–7.10 correspond to Section 5.

Note that our algorithm can compute $O(1)$ maximum flows: This uses $O(\min\{k, n \log (n^2/m)\}\, m)$ time and so is within our desired time bounds.

7.1 Poset representation

We implement the basic operations on the family $\mathcal{I}$ using the technique of [7], which we briefly summarize. Let $\mathcal{F}$ be an intersecting family over V, with every element in some $\mathcal{F}$-set. We will define a poset $\mathcal{P}$ that represents the minimal $\mathcal{F}$-set containing each element. (Note that throughout Section 7 $\mathcal{P}$ refers to the poset representation. So there is no confusion with the color partition $\mathcal{P}$.)

For any $x \in V$ let $M(x)$ denote the inclusionwise minimal $\mathcal{F}$-set containing x. Let $[x] = \{y : M(y) = M(x)\}$. Define a poset $\mathcal{P}$ whose elements are the distinct sets $[x]$, with $[y] \preceq [x]$ when $M(y) \subseteq M(x)$ (equivalently, when $y \in M(x)$). Note that $M(x) = \bigcup\{y : [y] \preceq [x]\}$.

An example is the representation of Picard and Queyranne of the s,t-mincuts [11]: Suppose a maximum s,t-flow has value k. For $x \in V$, $M(x)$ is the smallest set of in-degree k containing x and t but not s ($M(x) = V$ if no such mincut exists). A dag representation of the poset $\mathcal{P}$ can be explicitly constructed in $O(m)$ time given a maximum flow or preflow.

Consider a k-edge-connected digraph $G = (V, E)$. For any nonempty set $S \subseteq V$ let $\mathcal{I}^S$ ($\mathcal{O}^S$) be the family of sets that have in-degree (out-degree) k and are disjoint from S. (Take the ground set of both families to be V and add V to the family, so every vertex is in some set of the family.) $\mathcal{I}^S$ and $\mathcal{O}^S$ are intersecting families and have corresponding posets $\mathcal{P}$, which we denote as $\mathcal{P}(\mathcal{I}^S)$ and $\mathcal{P}(\mathcal{O}^S)$ respectively. [7] presents algorithms for the following operations on these two posets $\mathcal{P}$:

(a) Find the partition of V into sets $[x]$.
(b) Label every poset element $[x]$ with a minimal element $[y]$ of $\mathcal{P}$ satisfying $[y] \preceq [x]$.
(c) Mark every $[x]$ that has only one possible label $[y]$ in (b).

Note that operation (b) identifies the minimal elements of $\mathcal{P}$. Also note that in $\mathcal{I}^S$, $[S]$ consists of all vertices of V that do not belong to a set that has in-degree k and is disjoint from S.

Now we discuss the time bound for these operations. Recall that [6] finds the edge connectivity k of a given unweighted digraph in time $O(km \log n)$. Given the output of that algorithm, operations (a)–(c) each use time $O(m)$, in an unweighted digraph.

For weighted digraphs [10] finds the edge connectivity in time $O(nm \log (n^2/m))$. It is straightforward to modify this algorithm to perform operations (a)–(c) within the same time bound. We give a brief sketch for readers familiar with [10]. The sketch uses $O(nm)$ extra time and $O(n^2)$ space; slightly more care reduces the space to $O(m)$.

Let S be a nonempty set of vertices and let k be the edge connectivity. To implement the poset operations on $\mathcal{P} = \mathcal{P}(\mathcal{I}^S)$ begin by executing the Hao-Orlin algorithm with S as the initial set of source vertices. Number the preflows of value k that are found by the algorithm from 1 to r ($r < n$) in the order that these preflows are found. For each such preflow construct the Picard-Queyranne representation, calling it $\mathcal{Q}_i, i = 1, \ldots, r$. For any $x \in V$ the set $M(x)$ is defined both in our desired poset $\mathcal{P}$ and in the posets $\mathcal{Q}_i$; denote these sets as $M(x, \mathcal{P})$ and $M(x, \mathcal{Q}_i)$ respectively. Then $M(x, \mathcal{P})$ equals $M(x, \mathcal{Q}_i)$ where i is chosen as the largest index with $M(x, \mathcal{Q}_i) \neq V$ (if no such index exists then $M(x, \mathcal{P}) = V$). Using this observation construct each set $M(x, \mathcal{P})$ explicitly, by searching the appropriate poset $\mathcal{Q}_i$. Given all the sets $M(x, \mathcal{P})$ it is a simple matter to do operations (a)–(c) in time proportional to the total size of these sets, $O(n^2)$.

7.2 Derived families

Our algorithm modifies the given family $\mathcal{F} = (\mathcal{I}, \mathcal{O})$ using two operations, creating an $\mathcal{I}_B$-core (defined before Lemma 3.5) and making a core into a singleton (defined at the start of Section 5). The total effect of these operations on the family $\mathcal{I}$ is summarized as follows: Some elements x_i are duplicated d_i times, $i = 1, \ldots, a$. Some elements y_i, $i = 1, \ldots, b$ are made into singleton $\mathcal{I}$-sets. (Some of these y_i may be duplicates.) Some sets $V - z_i$, $i = 1, \ldots, c$ are made into $\mathcal{I}$-sets. As remarked in the discussion of the two operations, no set $V - z_i$ is a core.

Consider any family $\mathcal{I}^S$. The sets $M(x)$ get modified by these operations as follows. A set $M(y_i)$ changes to $\{y_i\}$. If $S = \{z_i\}$ for some i then $M(x)$ changes to $V - z_i$ for any element $x \notin S$ that originally had $M(x) = V$.

We modify the poset operations (a)–(c) so they work for the new family, as follows. Simple bookkeeping keeps track of duplicate elements.

(a) For $i = 1, \ldots, b$, remove y_i from its original poset element $[x]$ and create a new poset element $[y_i] = \{y_i\}$. If $S = \{z_i\}$ for some i then split $[S]$ into two elements, one equal to S and the other consisting of the remaining vertices.

(b) Each $[y_i]$ is its own label. An original label $[y]$ is still valid if it corresponds to a set that does not contain any y_i. In the opposite case choose some y_i ($i = 1, \ldots, b$) in the set and use $[y_i]$ as the new label.

(c) Mark each $[y_i]$. Now we show when to mark the remaining elements $[x]$. All notation (e.g., $\mathcal{I}$-core, $M(x)$) refers to the orginal family before it gets changed. Let $F = \{y_i : i = 1, \ldots, b,\ y_i$ does not belong to any $\mathcal{I}$-core$\}$. $[x]$ should be marked if each of the following conditions holds: (i) $[x]$ was marked in the original family; in that case let $[y] \preceq [x]$ be the corresponding label; (ii) at most one y_i, $i = 1, \ldots, b$ belongs to $[y]$; (iii) $M(x) \cap F = \emptyset$. Note that condition (iii)

holds if and only if $[x] \neq V$ in the poset $\mathcal{P}(\mathcal{I}^{S \cup F})$. So we can detect condition (iii) using operation (a).

7.3 Cores and zones

We give an algorithm that finds $\|\mathcal{I}\|$ $\mathcal{I}$-cores. This amounts to all the $\mathcal{I}$-cores when $\|\mathcal{I}\| > 1$.

Let S be an $\mathcal{I}$-core. (We show how to find S below.) Find the minimal elements of $\mathcal{P}(\mathcal{I}^S)$. If V is not minimal then these sets are the $\mathcal{I}$-cores disjoint from S. So these sets plus S are the desired $\mathcal{I}$-cores. If V is the unique minimal element then S alone is the desired $\mathcal{I}$-core.

The $\mathcal{I}$-core S can be found in several ways. The Hao-Orlin algorithm gives S directly. In general start with an arbitrary vertex x. If $\mathcal{I}^x$ is nontrivial (i.e., $\mathcal{I}^x \neq \{V\}$) then a core in $\mathcal{P}(\mathcal{I}^x)$ can be chosen as S. If $\mathcal{I}^x$ is trivial, the poset operations on $\mathcal{O}^x$ allow us to find a y with $\mathcal{I}^y$ nontrivial and we can proceed as above.

This algorithm allows us to calculate Φ when $\|\mathcal{I}\|, \|\mathcal{O}\| > 1$, since in that case all cores are known. If $\|\mathcal{I}\| = 1$ we still need to determine $\|\mathcal{I}_B\|$. Check if there is a black $\mathcal{I}$-core by choosing a white element w and checking $\mathcal{P}(\mathcal{I}^w)$ for a core.

We turn to finding the zones when $\|\mathcal{I}\| > 1$. Perform operations (b) and (c) on $\mathcal{P}(\mathcal{I}^S)$ (for S chosen as above). For an $\mathcal{I}$-core $I \neq S$, $\widehat{I}$ is the union of all sets $M(x)$ that contain I and no other $\mathcal{I}$-core. (This characterization would fail if instead of $\mathcal{I}^S$ we used $\mathcal{I}^s$ for some $s \in S$.) Hence, if I corresponds to the element $[y]$ in $\mathcal{P}(\mathcal{I}^S)$, $\widehat{I}$ consists of all elements x where $[x] \neq [S]$ has $[y]$ as its unique label. We can find $\widehat{S}$ the same way.

Our desired time bound precludes using this algorithm to recompute zones each time an edge is added to the cover. Instead we use a technique of [7]. Fix an intersecting family of sets $\mathcal{F}$, as above. An element x *elicits* an $\mathcal{F}$-core Y if $Y \subseteq M(x)$, i.e., any $\mathcal{F}$-set containing x contains Y. It is easy to verify the following properties in the intersecting family $\mathcal{I}^S$ [7]: If operation (b) labels $[x]$ with $[y]$ then x elicits the core corresponding to $[y]$. If x elicits Y then $\widehat{Y}$ and $\widehat{S}$ are the only zones that can contain x. ($\widehat{S}$ accounts for the possibility that $x \in [S]$.) Suppose we add an edge that keeps S an $\mathcal{I}_u$-core. Using the family $\mathcal{I}^S$ in the new family $\mathcal{F}_u$, if Y is still an $\mathcal{I}^S$-core then x still elicits Y.

7.4 Families with $\|\mathcal{I}\| = 1$: Lemma 2.9

If $\|\mathcal{O}\| \leq 2$ the algorithm follows the proof of Lemma 2.9, recomputing the zones each time an edge is added. Suppose $\|\mathcal{O}\| > 2$. Let O be the $\mathcal{O}$-core chosen in the proof. The proof adds an edge from N to a vertex not in $\widehat{N}$, for every $\mathcal{O}$-core $N \neq O$. We show how to choose these edges efficiently.

Since O is disjoint from $\widehat{N}$, we can choose an edge from N to O if $N \cup O$ is mixed. Take all such edges. Redefine $\mathcal{F}$ to reference the family of uncovered sets ($\mathcal{F}_u$). Now every $\mathcal{O}$-core is a set of the same color, say black.

Choose a white vertex w. In the family $\mathcal{O}^O$ suppose w elicits Y. (Find Y using operation (b) in $\mathcal{P}(\mathcal{O}^O)$.) It is easy to see that we can add an edge from N to w for every $N \neq O, Y$.

7.5 Augmenting procedure

Fix an $\mathcal{I}$-core S and an $\mathcal{O}$-core T. Consider two subfamilies $\mathcal{J} \subseteq \mu(\mathcal{I}^S)$, $\mathcal{N} \subseteq \mu(\mathcal{O}^T)$, such that any $I \in \mathcal{J}$ and $O \in \mathcal{N}$ have $I \cup O$ mixed. We show how to execute a sequence of augments that makes $\min\{|\mathcal{J}_u|, |\mathcal{N}_u|\} \leq 2$, i.e., all but at most 2 cores of either $\mathcal{J}$ or $\mathcal{N}$ get covered. After initializing the data structure using our poset operations, the time to execute all the augments is $O(n)$. We use the following technique of [7].

Choose one "representative" vertex from each core of $\mathcal{N}$. We maintain a partition of these representatives into sets $E(I)$, $I \in \mathcal{J}$, such that any vertex in $E(I)$ elicits I (in $\mathcal{I}^S$). Similarly we maintain a partition of representatives of the cores of $\mathcal{J}$ into sets $E(O)$, $O \in \mathcal{N}$, such that any vertex in $E(O)$ elicits O (in $\mathcal{O}^S$). Each set $E(I)$, $E(O)$ is represented as a list of its constituent vertices. These lists are initialized using operation (b) on posets $\mathcal{P}(\mathcal{I}^S)$ and $\mathcal{P}(\mathcal{O}^T)$.

To perform an augment choose three vertices from any nonempty lists $E(O)$. This gives three relations, namely that the representative of I_i elicits O_i, $i = 1, 2, 3$. Similarly we get three other relations, that the representative of O'_i elicits I'_i, $i = 1, 2, 3$. Using these six relations, it is easy to see that we can choose four distinct sets $I, I' \in \mathcal{J}$, $O, O' \in \mathcal{O}$ such that the representative of I elicits O' and the representative of O elicits I'. Lemma 2.6 shows O, I is compatible. Augment O, I, say by adding edge xy, $x \in O$, $y \in I$. We need only update the E-lists for the new family. Delete the representatives of I and O from $E(O')$ and $E(I')$. Now the only E-lists that are not valid are $E(I)$ and $E(O)$. Add $E(I)$ to $E(I')$ and $E(O)$ to $E(O')$.

We show this is correct by proving that for any vertex $v \in E(I)$, any uncovered $\mathcal{I}^S$-set J containing v contains I'. $v \in E(I)$ implies $I \subseteq J$. Thus $y \in J$, and J uncovered implies $x \in J$. Since $x \in O$ Lemma 2.3 shows $O \subseteq J$ (compatibility implies $J \not\subseteq O$). Thus the representative of O belongs to J. Hence $I' \subseteq J$.

We use the above augmenting procedure to implement Corollary 2.11: Start by augmenting to make the number of $\mathcal{I}$-cores or the number of $\mathcal{O}$-cores 2 or less. If it is 2 perform one more augment by recomputing the zones. After that apply the procedure for Lemma 2.9.

7.6 Dangerous core procedure

We give a subroutine that is used in the next section to process singleton families. In a singleton family consider a sequence of augments y_i, x_i, $i = 1, \ldots, r$, where each y_i (x_i) is an $\mathcal{O}_W$- ($\mathcal{I}_B$)-core. We give a procedure that finds the first of these augments to create a 1-blocker, or else verify that no 1-blocker is ever created. The procedure runs in time $O(m)$ after finding a maximum flow. We describe how to find the first 1-blocker for W. The complement-symmetric procedure works for B.

We want to find the first augment that makes $\mathcal{O}[\mathcal{I}_W, \mathcal{O}_W] \neq \emptyset$. Let $\mathcal{O}_0$ be the set of all $\mathcal{O}_W$-cores that remain after the r augments are executed. We will maintain the set $\mu^*(\mathcal{O}[\mathcal{I}_W, \mathcal{O}_0])$ as edges $y_i x_i$ are added to the graph.

To initialize this set, form a new graph G' by starting with the graph without any of the edges $y_i x_i$, contracting every $\mathcal{I}_W$-core to a vertex S and contracting $\mathcal{O}_0$ to a vertex T. Find a maximum flow from S to T in G'. Assume the flow value is k, else there is no 1-blocker. The smallest mincut M (i.e., the inclusionwise minimal set of out-degree k containing S and disjoint from T) is the desired set $\mu^*(\mathcal{O}[\mathcal{I}_W, \mathcal{O}_0])$. Initialize a count c to be the number of $\mathcal{O}_W$-cores in M. $c = 0$ if and only if we have a 1-blocker for W.

Now add each edge $y_i x_i$ in turn to G', using the following procedure: If $y_i x_i$ leaves M then use the Ford-Fulkerson labelling procedure [1] to enlarge M to the smallest mincut in the new graph. Then update c, decreasing it by 1 since y_i is no longer an $\mathcal{O}_W$-core and increasing it by 1 for every $\mathcal{O}_W$-core that the labelling procedure adds to M. If $c = 0$ we have discovered the first 1-blocker for W; both y_i and x_i are dangerous. If not continue by adding the next edge.

7.7 Singleton families: Lemma 3.5

Suppose $\|\mathcal{I}_B\| = \|\mathcal{O}_W\| = 2$. We know there are two edges, each joining an $\mathcal{O}_W$-core to an $\mathcal{I}_B$-core, that give a family containing no $\mathcal{O}_W$- or $\mathcal{I}_B$-core. There are only 2 possibilities for the edges, so we can guess the edges and check our guess. Now we can assume $\|\mathcal{I}_B\| = \|\mathcal{O}_W\| = 0$. Use the augmenting procedure of Section 7.5 to augment all but at most two $\mathcal{I}_W$- and $\mathcal{O}_B$-cores. Then guess the last two edges, as above.

Now suppose $\|\mathcal{I}_B\| = \|\mathcal{O}_W\| > 2$. Use the augmenting procedure to augment all but 2 $\mathcal{I}_B$-cores. Then execute the dangerous core procedure (for both B and W). If the final graph is not a 1-blocker then proceed as in the previous paragraph.

In the opposite case the dangerous core procedure finds the dangerous cores y_i, x_i. Redo the augments as follows: First execute all augments preceding y_i, x_i. Then reexecute the augmenting procedure, with y_i and x_i now unavailable for augmenting. The procedure augments all but 3 $\mathcal{I}_B$-cores (one of which is x_i) without creating a 1-blocker (as in the proof of Lemma 3.5). Hence we can proceed as above.

7.8 Finding blockers

Lemma 4.2 shows we need only detect a nondegenerate blocker. For families $\mathcal{A}, \mathcal{B} \neq \emptyset$ we test if $\mathcal{O}[\mathcal{A}, \mathcal{B}] \neq \emptyset$ by computing a maximum flow in the graph with every vertex in a set of $\mathcal{A}$ ($\mathcal{B}$) contracted to a source (sink) vertex.

We turn to the procedures for Section 5. Implementing Lemmas 5.1–5.4 is straighforward. The sets X_B, I_B and Y_B are easily found by computing a maximum flow, assuming $\mathcal{I}_B, \mathcal{O}_B \neq \emptyset$.

7.9 Finding a good tuple

We begin with a useful fact. Assume $\mathcal{I}_B, \mathcal{O}_B, \mathcal{O}[\mathcal{I}_B, \mathcal{O}_B] \neq \emptyset$. Let $\mathcal{Q}$ be the poset representation of all sets of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$. More specifically to construct $\mathcal{Q}$, form a new graph by contracting every $\mathcal{I}_B$-core to form a new vertex S and contracting every $\mathcal{O}_B$-core to form a new vertex T. Find a maximum flow from S to T. $\mathcal{Q}$ is the Picard-Queyranne representation of the mincuts (i.e., the sets of out-degree k containing S and disjoint from T).

Lemma 7.1 *Any $\mathcal{O}$-core is contained in some element $[x]$ of poset $\mathcal{Q}$. Similarly any $\mathcal{I}$-core not containing X_B is contained in some element $[x]$ of $\mathcal{Q}$.*

Proof: Consider a set S that is not contained in any poset element $[x]$. S contains vertices y, z with $y \notin M(z)$. Thus some set of $\mathcal{O}[\mathcal{I}_B, \mathcal{O}_B]$ contains z but not y. S is not an $\mathcal{O}$-core by Lemma 2.5. If S is an $\mathcal{I}$-core then Lemma 2.3 implies $X_B \subseteq M(z) \subseteq S$. □

Now we show how to find a good tuple. Label the poset elements that contain an $\mathcal{O}$-core, and also those that contain an $\mathcal{I}$-core. (O, B) is good if O is contained in $[S]$ (i.e., $O \subseteq X_B$). (I, B) is good if an element of I belongs to $[T]$. Clearly we can find all such good tuples within our time bound.

Suppose there are no such good tuples. (I, O, B) is good if $[y] \leq [x]$ for poset elements $[y]$ containing O and $[x]$ containing an element of I. Check for this condition by doing a depth-first search from the elements $[x]$ labelled by $\mathcal{I}$-cores.

In what follows we refer to the preceding procedure to find a triplet (I, O, B) as the *good triplet procedure*. We use various modifications of this procedure. For instance we can restrict the triplets not to contain the $\mathcal{O}$-core O (by not assigning a label for O) or the $\mathcal{I}$-core I (by not starting a search from I). We can find all triplets involving O (by not using labels for any other $\mathcal{O}$-core) or I (by only searching from I).

7.10 Families with $\Phi = \|\mathcal{I}\| = \|\mathcal{O}\|$: Lemma 5.8

We show how to implement Lemma 5.8, omitting the straightfoward details. Assume the given family is not a blocker.

Case 1. $\mathcal{O}_W = \emptyset$.

We modify the procedure of the proof: Execute Step 1. Then execute Step 2 for just one set Z_I. The resulting family $\mathcal{F}'$ has $OPT = \Phi$ as proved in the lemma. Hence $\mathcal{F}'$ is not a blocker. Since $\mathcal{F}'$ has white $\mathcal{I}$- and $\mathcal{O}$-cores it can be treated by a subsequent case. This procedure runs within our time bound since we find Z_I by computing one maximum flow.

Case 3. $\|\mathcal{O}_X\| = 0$.

As noted in the previous section we can find all good tuples (I, B), (J, W) within our time bound. The rest of Case 3 is simple.

Claim 2:

We must find distinct mixed $\mathcal{O}$-cores $O \subseteq X_B$ and $N \subseteq I_W$, for some mixed $\mathcal{I}$-core I. If X_B contains two mixed $\mathcal{O}$-cores, use the good triplet procedure to find a good triplet involving W. In the opposite case X_B contains a unique mixed $\mathcal{O}$-core O. Find a good triplet containing W but not O by executing the good triplet procedure.

Case 6. Both colors B satisfy $\|\mathcal{I}_B\| + \|\mathcal{O}_B\| \leq \Phi - 2$.

We must find two disjoint good triplets (I, O, B), (J, N, W). Find a good triplet (I, O, B). Check if there is a good triplet disjoint from it by executing the good triplet procedure. Assume no such triplet exists, else we are done.

Now we can find all good triplets involving W by executing the good triplet procedure twice, to find all good triplets involving W and I and all good triplets involving W and O. Choose two distinct good triplets (I, O_i, W), $i = 1, 2$. (If two such do not exist, choose as many good triplets as possible.) Similarly choose distinct good triplets (I_i, O, W), $i = 1, 2$. One of these four triplets can be chosen as part of the desired pair of disjoint triplets. For each of the four triplets, use the good triplet procedure to check if B has a good triplet disjoint from it.

Theorem 7.2 *The bipartite digraph edge-connectivity incrementation problem can be solved in time $O(km \log n)$ for an unweighted digraph and $O(nm \log(n^2/m))$ for a weighted digraph. Here k is the target connectivity.* □

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