Rosenblatt's Preceptron Learning Algorithm

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Today's Lecture Goals

- Introduction to binary classification
- The linear discriminative classifier
- Rosenblatt's Preceptron Learning Algorithm
- Class projects

Binary Classification

- A binary classifier is a mapping from a set of *d* inputs to a single output which can take on one of TWO values
- In the most general setting

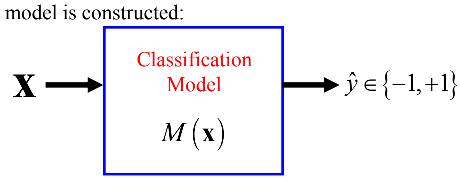
  inputs: $\mathbf{x} \in \mathbb{R}^d$

  output: $y \in \{-1, +1\}$
- Specifying the output classes as -1 and +1 is arbitrary!
  - Often done as a mathematical convenience

A Binary Classifier

Given learning data: $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)$

A model is constructed:



The Learning Data

- Learning algorithms don't care where the data comes from!
- Here is a toy example from robotics…
  - Inputs from two sonar sensors: sensor 1: $x_1 \in \mathbb{R}$ sensor 2: $x_2 \in \mathbb{R}$
  - Classification output:
    - Robot in Greg's office: y = +1
    - Robot NOT in Greg's office: y = -1

The Learning Data

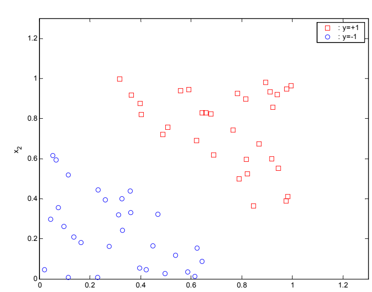
	x_1	x_2	y
Example 1	0.95013	0.58279	1
Example 2	0.23114	0.4235	-1
Example 3	0.8913	0.43291	1
Example 4	0.018504	0.76037	-1
…	…	…	…

The Learning Data

- Symbolic Representation of *N* learning examples of *d* dimensional inputs

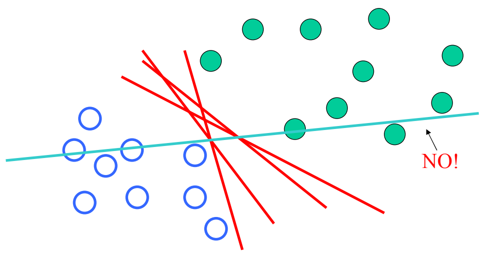
$$\begin{pmatrix} x_{11} & \cdots & x_{1d} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{N1} & \cdots & x_{Nd} & y_N \end{pmatrix}$$

Graphical Representation of Training Data

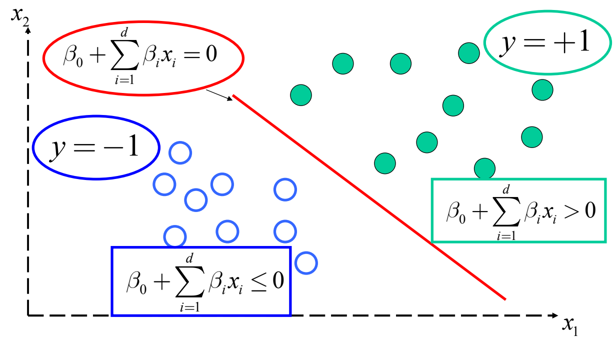


Linear Separating Hyper-Planes

How many lines can separate these points?



Linear Separating Hyper-Planes



Linear Separating Hyper-Planes

- The Model:

$$\hat{y} = M(\mathbf{x}) = \operatorname{sgn}\left[\hat{\beta}_0 + \left(\hat{\beta}_1, \ldots, \hat{\beta}_d\right)\mathbf{x}^T\right]$$

- Where:

$$\operatorname{sgn}[A] = \begin{cases} 1 & \text{if } A > 0 \\ -1 & \text{otherwise} \end{cases}$$

- The decision boundary:

$$\hat{\beta}_0 + \left(\hat{\beta}_1, \ldots, \hat{\beta}_d\right)\mathbf{x}^T = 0$$

Linear Separating Hyper-Planes

- The model parameters are:

$$\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right)$$

- The *hat* on the betas means that they are estimated from the data
  - In the class notes… Sometimes the hat will be there and sometimes it won't!
- Many different learning algorithms have been proposed for determining $\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right)$

Rosenblatt's Preceptron Learning Algorithm

- Dates back to the 1950's and is the motivation behind Neural Networks
- The algorithm:
  - Start with a random hyperplane $\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right)$
  - Incrementally modify the hyperplane such that points that are misclassified move closer to the correct side of the boundary
  - Stop when all learning examples are correctly classified

Rosenblatt's Preceptron Learning Algorithm

- The algorithm is based on the following property:
  - Signed distance of any point $\mathbf{x}$ to the boundary is *proportional* to $\hat{\beta}_0 + \left(\hat{\beta}_1, \ldots, \hat{\beta}_d\right)\mathbf{x}^T$
- Therefore, if M is the set of misclassified learning examples, we can push them closer to the boundary by minimizing the following

$$D\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right) = -\sum_{i \in M} y_i \left(\hat{\beta}_0 + \left(\hat{\beta}_1, \ldots, \hat{\beta}_d\right)\mathbf{x}_i^T\right)$$

Rosenblatt's Minimization Function

- This is classic Machine Learning!
- First define a cost function in model parameter space

$$D\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right) = -\sum_{i \in M} y_i \left(\hat{\beta}_0 + \sum_{k=1}^{d} \hat{\beta}_k x_{ik}\right)$$

- Then find an algorithm that modifies $\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right)$ such that this cost function is minimized
- One such algorithm is **Gradient Descent**

The Gradient Descent Algorithm

$$\hat{\beta}_i \leftarrow \hat{\beta}_i - \rho \frac{\partial D\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d\right)}{\partial \hat{\beta}_i}$$

Where the learning rate is defined by: $\rho > 0$

The Gradient Descent Algorithm for the Perceptron

$$\frac{\partial D(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d)}{\partial \hat{\beta}_0} = -\sum_{i \in M} y_i \qquad \frac{\partial D(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d)}{\partial \hat{\beta}_j} = -\sum_{i \in M} y_i x_{ij}, \qquad j = 1, \ldots, d$$

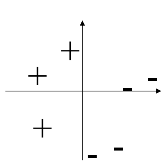
$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_d \end{pmatrix} \leftarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_d \end{pmatrix} - \rho \begin{pmatrix} y_i \\ y_i x_{i1} \\ \vdots \\ y_i x_{id} \end{pmatrix}$$

The Good Theoretical Properties of the Perceptron Algorithm

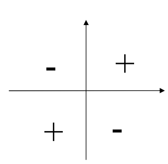
- If a solution exists the algorithm will always converge in a finite number of steps!
- Question: Does a solution always exist?

Linearly Separable Data

- Which of these datasets are separable by a linear boundary?



a)



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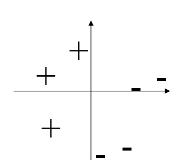
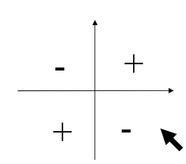
Linearly Separable Data

- Which of these datasets are separable by a linear boundary?



a)



b)

Not Linearly Separable!

Bad Theoretical Properties of the Perceptron Algorithm

- If the data is not linearly separable, algorithm cycles forever!
  - Cannot converge!
  - This property stopped research in this area between 1968 and 1984…
    - *Perceptrons*, Minsky and Pappert, 1969
- There are infinitely many solutions
- When data is linearly separable, the number of steps to converge can be very large (depends on size of gap between classes)

What about Nonlinear Data?

- Data that is not linearly separable is called nonlinear data
- Nonlinear data can often be mapped into a nonlinear space where it is linearly separable

Nonlinear Models

- The Linear Model:

$$\hat{y} = M(\mathbf{x}) = \text{sgn}\left[\hat{\beta}_0 + \sum_{i=1}^{d} \hat{\beta}_i x_i\right]$$

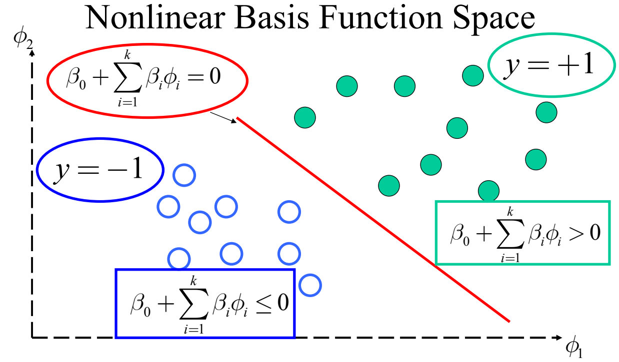
- The Nonlinear (basis function) Model:

$$\hat{y} = M(\mathbf{x}) = \text{sgn}\left[\hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i \phi_i(\mathbf{x})\right]$$

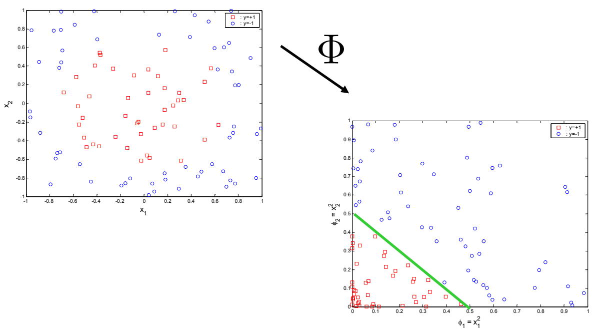
- Examples of Nonlinear Basis Functions:

$$\phi_1(\mathbf{x}) = x_1^2 \quad \phi_2(\mathbf{x}) = x_2^2 \quad \phi_3(\mathbf{x}) = x_1 x_2 \quad \phi_4(\mathbf{x}) = \sin(x_{55})$$

Linear Separating Hyper-Planes In Nonlinear Basis Function Space



An Example



Picking a Model Structure?

- How do you pick the basis functions?
  - The number and type?
- These are called **learning parameters**
  - Two approaches choosing learning paramters
    - Bayesian
      - Learning parameters must maximize probability of correct classification based on prior biases
    - Frequentist
      - Use validation data
- More on learning parameter selection later

Perceptron Algorithm Convergence

- Two problems:
  - No convergence when data is not separable in basis function space
  - Gives infinitely many solutions when data is separable
- Can we modify the algorithm to fix these problems?
- See Homework 1 (next week)….