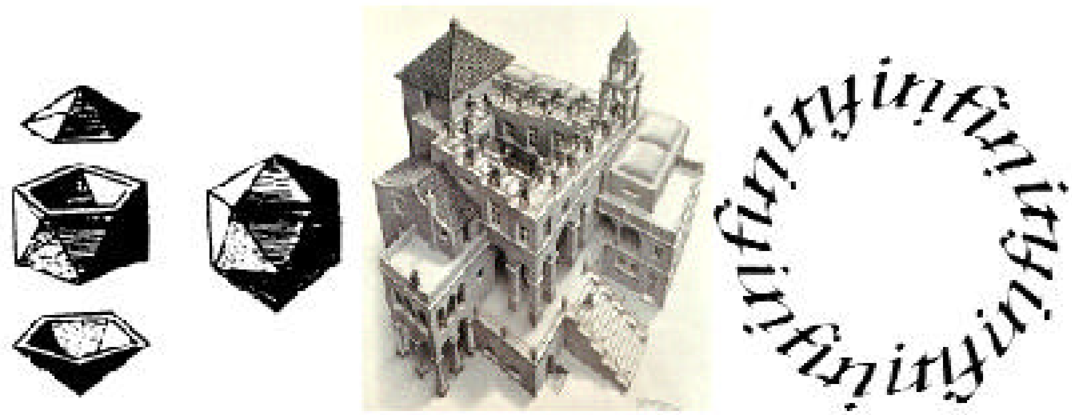
2. Tools for Spatial Thinking

2.1 Some Definitions of Spatial Ability



(a) (b) (c)

Figure 2-1. Samples of work by (a) Johannes Kepler, (b) M.C. Escher, and (c) Scott Kim.

The figure above shows examples of work by extraordinary spatial thinkers: Johannes Kepler's diagram (from his book *Harmonices Mundi*, 1619) shows an imaginative parsing of the icosahedron into an antiprism and two pyramids; M. C. Escher's *Ascending and Descending* (1960) is a brilliant example of optical illusion; and the contemporary puzzlesmith Scott Kim (1981) has designed a series of planar "inversions" playing off the symmetry of letter forms.

When we read geometric writings by someone like Kepler, or when we view creations by artists like Escher and Kim, we somehow sense that we are looking at the work of great spatial thinkers. But this is only a matter of informal intuition. What does it mean to think spatially? Clements and Battista (1992) define spatial reasoning as consisting of "cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated." Linn and Peterson (1985) similarly state that spatial reasoning "refers to skill in representing, transforming, generating, and recalling symbolic, nonlinguistic information". (p. 1482).

Historically, L.L. Thurstone argued for both the existence and independence of spatial intelligence and characterized it as one of seven factors of intellect (Gardner, 1983, p. 175). Moreover, according to Thurstone, spatial ability itself can be further divided into three skills: recognizing an object from different angles, imagining movement or displacement of internal parts of a spatial configuration, and determining spatial relationships with respect to one's own body. The divisions between these categories are the subject of much debate and a large number of classification schemes exist for different types of spatial skills. (Cf. Borich and Bauman, 1972; Linn and Peterson, 1985). What follows is an attempt to provide examples of different kinds of spatial thinking without attempting to resolve all of the scholarly debates surrounding the subject.

A classic example of object recognition from different angles is shown in Figure 2-2 below. The figure contains a sample diagram from Shepard and Metzler's (1971) experiment in mental rotation in which participants were asked to indicate whether one form in the pair was a rotation of the other. In this case, the two objects are in fact two different views of the same three-dimensional block figure (here, rotated around an axis perpendicular to the plane of the paper); other pairs of objects in Shepard and Metzler's original experiment were in fact distinct (non-superimposable) objects.

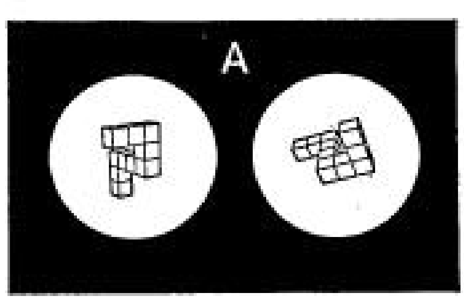


Figure 2-2. A sample mental rotations task. (Shepard & Metzler, 1971).

Other tasks of mental rotation include the Flags and Cards Test (French, Ekstroms, & Price, 1963) and the Primary Mental Abilities test (Thurstone & Thurstone, 1941).

An example of mental movement of spatially presented information is shown in the mental paper folding tasks pictured in Figure 2-3 (Shepard and Feng, 1972) in which subjects are asked to indicate whether the arrows will meet when the folding net of the cube is assembled:

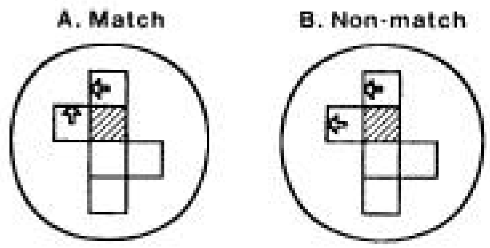


Figure 2-3. Samples of mental paper-folding tasks. (Shepard & Feng, 1972).

For other examples of multistep visualization tasks, see also: the Wechsler Intelligence Scale for Children-Revised [WISC-R], the Wechsler Adult Intelligence Scale [WAIS], and the Differential Aptitude Test [DAT].

The scenario below illustrates a sample task involving the determination of spatial relationships with respect to one's own body.

> ... Guilford and Zimmerman (1956) developed a spatial orientation test which required the subject to examine carefully five pictures which would show [a] prow of [a] boat and to select the one picture which matches a schematic drawing of a line and a dot representing the horizon and the point of the prow.
> (Vandenburg, 1975, p. 59).

Tasks similar to this one in which orientation relative to one's body plays a key role also include the Rod and Frame Test (Witkin, Dyk, & Faterson, 1962) and the water-level task (Inhelder & Piaget, 1958).

Returning to the larger picture, spatial intelligence as one of Gardner's multiple intelligences (Gardner, 1983) might be characterized as a discrete kind of intellect. It is one of a number of "sets of know-how" (p. 69) -- an intelligence involved with objects. It is "tied fundamentally to the concrete world, to the world of objects and their location in the world." (p. 204). The spatial learning explored in this thesis is tied to specific objects -- polyhedra -- and concerns the mental folding and unfolding of two-dimensional folding nets into their corresponding three-dimensional mathematical forms. The link between mathematical objects and spatial ability is one to which we will continually return in this thesis.

2.2 Spatial Ability and Science/Mathematics Achievement

Having come this far in our discussion of spatial thinking, it might be argued that this type of cognition could be unconnected to less clearly spatial pursuits. It would be interesting enough if spatial thinking were of merely artistic value in the creation of the types of work typified by Escher and Kim. But in fact, those interested in science and mathematics education likewise have reason to pay attention to the study of spatial thinking.

Many studies have indicated a strong correlation between high spatial visualization skills and success in science and mathematics. The findings of Siemankowski and Macknight (1971) "show there is strong evidence that science majors differ from nonscience-oriented students in three-dimensional conceptualization." (p. 59). Similarly for mathematics education, Guay and McDaniel (1977) have found that "...among elementary school children, high mathematics achievers have greater spatial ability than low mathematics achievers." (p. 214). Skolnick, Langbort, and Day (1982) argue that spatial ability plays an important role in children's understanding in mathematical and scientific concepts. Poole and Stanley (1972) conclude that spatial visualization is an important factor for predicting the success of first year engineering students. Other results indicating a correlation between spatial ability and science/mathematics achievement are reported in (Bishop, 1973), (Pallrand and Seeber, 1984), and (Mitchelmore, 1980a).

Among possible reasons for this correlation is the need to employ spatial thinking when approaching such tasks as approximating the magnitude of a figure, determining mathematical relationships, or mentally moving or assessing the size and shape of part of a figure. (Tartre, 1990). Three-dimensional relations are also important when the problem involves rendering a three-dimensional situation in a two-dimensional diagram "which is used both to suggest appropriate computational procedures and to communicate results." (Mitchelmore, 1980). In the science classroom Siemankowski and McKnight (1971) suggest some specific examples:

> [S]cience students are constantly subjected to diagrams, usually of two dimensional representations of three dimensional models ... The need for three dimensional conceptualization is necessary for the comprehension of wave energy transmission, chemical bonding, fields of force, structure of the atom, x-ray diffraction patterns, DNA, cell division, and countless other concepts and phenomena found in every branch of science. (p. 56).

Focusing on mathematics education, Usiskin (quoted in Clements, 1987) has described four dimensions, three of which require spatial reasoning:

> (a) visualization, drawing, and construction of figures; (b) study of the spatial aspects of the physical world; (c) use as a vehicle for representing nonvisual mathematical concepts and relationships; and (d) representation as a formal mathematical system. (p. 420).

Moreover, spatial visualization skills may in general indicate "a particular way of organizing thought in which new information is linked to previous knowledge structures to help make sense of the new material." (Tartre, 1990).

2.3 Is Spatial Thinking Trainable? The Case for Manipulatives

> Spatial thinking ... has been suggested by famous mathematicians such as Hadamard and Einstein to be essential to creative thought in all high level mathematics ... Given their importances ... it is essential that geometry and spatial reasoning receive greater attention in instruction and in research.
>
> (Clements and Battista, 1992, p. 457).

Given that ability in spatial reasoning is important in mathematics and science education, it is natural to wonder whether these kinds of skills can be taught. Various studies have shown that spatial ability can be improved through training (cf. Ben-Chaim et al., 1988 and Battista et al., 1982), but of special interest to mathematics and science educators is that the use of manipulative materials plays a central role in many types of spatial learning experiences.

A study by Brinkmann (1966) has shown that students participating in programmed-instruction stressing concrete object manipulation in spatial visualization activities scored significantly higher than a control group on visualization post-tests. Bishop's (1973) study also found that students enrolled at schools where teaching was "based very firmly on the use of structural apparatus" (p. 43) outscored their peers on standardized tests of spatial thinking, and that mere informal handling of 3D shapes may promote growth in spatial ability.

In a similar fashion, Lord (1985) used a "planes through solid" intervention in which students were required to picture the cross-sectional slice of a three-dimensional shape and predict the two-dimensional shape of the cut surface. The experimental group in this study used geometrical manipulatives for 12 weeks of exercises in visualizing cross-sections. Post-tests against a control group which did not take part in the intervention showed "significant posttest difference were found for the experimental population, but not for the nonexperimental group." (p. 401)

Even in the context of more free-form play, Serbin and Connor (1979) found a correlation between play with toys such as blocks, Lincoln Logs, Tinker Toys, Lego blocks, and Erector Sets -- that is, toys that encourage manipulation, construction, or movement through space (Mitchelmore, 1973) -- and visual-spatial performance:

> These kinds of toys provide users with concrete experiences in the manipulation of objects, patterns, construction, and movement through space. More importantly, these toys must be manipulated so that they 'work.' (Tracy, 1987, p. 125).

In a cross-cultural study, Mitchelmore (1980b) found British students to be developmentally three years ahead of their counterparts in the United States in spatial and drawing ability, and suggested that this may be due in part to differences in the way that manipulatives are used in the classroom. In the British classrooms he observed "tables piled high with boxes and tins of all shapes and sizes ..." (p. 212) with no such collections in the American schools.

Even more forcefully than suggesting that spatial visualization improves with experience with manipulatives, Ben-Chaim (1989b) has concluded:

> ... there is evidence that *unless* adolescent students have concrete and semi-concrete experiences (building with cubes, representing 3-dimensional objects in 2-dimensional drawings, and reading such drawing), most of them will have difficulties in "seeing" hidden parts ... (p. 58, emphasis added).

Ben-Chaim's conclusion is certainly consistent with the findings of Piaget and Inhelder (1941) for younger children in which they asked the children to draw what a paper shape would look like if it was unfolded and placed flat on the table in front of them:

> ... imagining the rotation and development of surfaces depends largely on the actual process of unfolding solids, and the motor skills involved in such actions. In particular, the child who is familiar with folding and unfolding paper shapes through his work at school is two or three years in advance of children who lack this experience. (p. 276).

2.4 Beyond Manipulatives

2.4.1 Why Combine Manipulatives and Software?

Fewer studies have documented the effectiveness of software in promoting spatial thinking. McClurg and Chaille (1987) explore computer environments as possible means by which to enhance children's spatial thinking skills; there have been studies in the effectiveness of LOGO environments in promoting spatial thinking (cf. Sgroi, 1990; Clements and Battista, 1992); yet a gap exists in exploring the way that spatial thinking can be enhanced by a *combination* of software and real-world activities with mathematical manipulatives.

While polyhedral building kits such as Polydron (M2), Zome Tool (M3), and Googolplex(M1) are elegantly designed mathematical manipulatives, a real-time software environment offers possibilities that purely physical (non-computational) manipulatives cannot. A software environment might, for example, permit children to study the effect of transformations -- such as stretching, slicing, and truncation -- on both the objects and their folding nets, something not possible with static materials. Children in this way can explore their own online "algebra of manipulatives". Conversely, manipulatives offer a tactile element that typical "virtual" (screen-based) software applications cannot. The two systems to be discussed in this thesis -- HyperGami and JavaGami -- are attempts to effect this integration of the strengths of both manipulatives and software.

2.4.2 Personalization of Mathematics and Dignified Educational Activity

Something often overlooked in the development of educational materials, whether teaching science or geometry or spatial thinking, is the value of enabling children to personalize their educational experiences. HyperGami and JavaGami are designed with a strong eye toward this issue. In these software environments, children can personalize their shapes not only in the three-dimensional modeling stage, but also when they are coloring, doodling, and drawing shapes on the folding nets. When they print the nets and assemble them with a variety of materials (including glue, scissors, tape) and even festoon their creations with stickers, googly eyes, and glitter, they have been engaged in activity with tangible objects and have created their own manipulative material (which can in turn be used for other manipulative-based activities). The end results of working with the software are personalized manipulatives which children have given as gifts to parents, friends, and teachers. Children have given names to their models (such as "Fred" for a dodecahedron); they have hung the objects on their bedroom ceilings; displayed them on the family mantle; adorned the Christmas tree during the holidays; and used them as gift boxes for special presents.

Just as important as the affective significance of the end-products is children's enjoyment of the activity itself. These are software systems used by both children and adults -- distribution of HyperGami and JavaGami has included sculptors, jewelers, polyhedral model builders, and professional mathematicians -- and when children use a real system for creating real things, they take pride in the activity.

On a broader scale, working with polyhedra is a dignified mathematical activity that a child may carry into adulthood while his ideas and level of mathematical sophistication continue to grow. By working with polyhedra, children become linked with objects rich with mathematical meaning and history: models of dodecahedra cut from soapstone have been dated as far back as 500 BC (Artmann, 1993); Plato wrote about the five regular solids in the Timaeus around 400 BC (Senechal, 1988); they were used by Kepler in his conjecture about planetary orbits (Senechal, 1988); and paper polyhedral models built and used by Kästner in 1780 for mathematics teaching are still on display at the University of Göttingen today (Mühlhausen, 1993). Jean Pedersen (1988) feels that the tangible mathematical nature of polyhedral models themselves is an important factor in motivating students:

> When I talk to students about polyhedra, they don't ask, 'Why do we have to study such things?' ... The first reason for studying polyhedra is that they are tangible. This is particularly important to secondary students who do a lot of things that are neither mathematical nor tangible. (p. 133).

Modern-day mathematicians including H.S.M. Coxeter, Magnus Wenninger, and John Conway have made the study of polyhedra an important part of their life's work. Conway was in fact recently photographed by the New York Times sitting in his office -- an office overflowing with polyhedra he had constructed. By seeing past and present adult role models engaged in the same kind of mathematics that they are learning, children see that the activities they are doing are not just for kids but have a component in the world of adults as well. This is an important part of creating dignified activities and software for children -- it is important for educational activities to be accessible and to provide engagement for children at different levels of mathematical maturity without being patronizing or condescending.