Orihedra: Mathematical Sculptures in Paper[1]

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Abstract

Mathematics, as a subject dealing with abstract concepts, poses a special challenge for educators. In students' experience, the subject is often associated with (potentially) unflattering adjectives—"austere", "remote", "depersonalized", and so forth. This paper describes a computer program named HyperGami whose purpose is to alleviate this harsh portrait of the mathematical enterprise. HyperGami is a system for the construction of decorated paper polyhedral shapes; these shapes may be combined into larger polyhedral sculptures, which we have dubbed "orihedra." In this paper, we illustrate the methods by which orihedra may be created from HyperGami solids (using the construction of a particular sculpture as an example); we describe our experiences with elementary- and middle-school students using HyperGami to create orihedra; we discuss the current limitations of HyperGami as a sculptural medium; and we outline potential directions for future research and software development.

1. Introduction: Finding a Path Toward Personalized Expression in Mathematics

Mathematics is, undeniably, a discipline largely devoted to the study of abstract concepts: numbers, functions, groups, *n*-dimensional vector spaces (to mention a few examples) all have a quality of unreality—or perhaps "otherworldliness"—about them. For working mathematicians, this abstract quality comes with the territory of the discipline—as suggested by Bertrand Russell's often-quoted aphorism, "Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true"[2]. Russell's quote was motivated by his work in the (perhaps maximally abstract) world of mathematical logic; but even in that most visual branch of mathematics, classical geometry, the objects of study transcend everyday experience. The first book of Euclid's *Elements* [Heath, 1925] presents us with objects that, from the outset, practically defy comprehension:

1. A point is that which has no part.
2. A line is breadthless length...
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.

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[1] A much-condensed version of this paper appeared in the *Proceedings of the 1996 National Educational Computing Conference*.

[2] Quoted in [Trudeau, 1987], p. 164.

This abstract nature of mathematics poses a dilemma for educators. On the one hand, while mathematical objects are intended to transcend experience, students must nonetheless arrive at these abstractions through the objects and insights of worldly experience—often a difficult intellectual process. Perhaps more important than this *cognitive* issue, however, is the associated *affective* (or emotional) issue. Mathematics educators often despair at the "austere" or "cold" reputation of their subject , and wonder how they can present the subject in such a way as to render it more meaningful, important, or personalized to students.

It is our belief that such a personalized representation of mathematics is in fact essential to the study of the discipline. In support of this claim, it must be noted that even among those who love mathematics, the discipline hardly speaks with one "abstract" voice. For many mathematically trained professionals—architects, sculptors, designers, engineers—the subject is reflected, first and foremost, in the creation of tangible objects; and while these professionals may indeed acknowledge the abstract quality of the mathematical concepts they employ, they still take an undiluted pleasure in making these concepts manifest in the physical world. Moreover, even theoretical mathematicians often speak of the need to achieve an understanding of abstract objects through conceptions whose origins are visible or concrete. As Hilbert writes:

> "In mathematics... we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied... On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations..." (In [Hilbert and Cohn-Vossen, 1932], p. iii.)

How, then, is mathematics education to be rendered more "personalized"? This paper reports on a computational system named *HyperGami*—an application for the creation of polyhedral models and sculptures—that reflects one style of response to this question. In effect, HyperGami is a system that weaves together various themes in mathematics education—the cognitive role of mathematical manipulatives as tangible representatives of abstract concepts [cf. Welchman-Tischler, 1992]; the "constructionist" approach to learning as mediated through the creation of artifacts [Papert, 1991]; the social or emotional role that physical objects are capable of playing in students' lives [Eisenberg and DiBiase, 1996]; and the creative benefits of computational media (with an emphasis on the use of "domain-enriched" language environments [Eisenberg, 1991]). Most importantly, HyperGami reflects an approach to learning mathematics through an expressive and dignified activity in which students can take part; rather than presenting students with mathematical training intended to be cashed in at some distant time and for some unknown task, the system allows the domain of mathematics itself to act as its own best advocate, here and now.

It should be noted that HyperGami does not, emphatically, represent a *universal* or *surefire* response to the challenge of rendering mathematics education more meaningful to students. The domain of polyhedral modelling is an idiosyncratic one, possessed (as we have gradually learned) of its own gleefully quirky subculture. Not every student will wish to approach mathematics by this route, nor should they. Nonetheless, we see HyperGami as one plausible instance of a style of mathematics education that focuses on the development of (for lack of a better term) "musical instruments" for mathematics: tools that support long-term growth and individual craftsmanship in the service of a mathematically rich vocation.

Indeed—and just to pursue the ideas of the previous paragraph—the analogy with musical instruments is, we feel, a productive one. Any individual musical instrument is adopted by at most a small number of students in a given classroom or school. Still, a significant percentage of students may find themselves adopting *some* musical instrument in the course of their education; and for those students for whom the instrument becomes a passion, their engagement with that instrument can form the basis of a lifetime of intellectual growth and creativity. Moreover, musical instruments act as the foci of their own individual subcultures of practitioners; and a body of lore evolves around the use of any one instrument. In the same vein, we believe that a large and varied sampling of mathematical "instruments" (of which HyperGami is just one instance) may be capable not only of representing mathematical ideas but of fostering passionate and eccentric mathematical communities across a wide range of ages.

The remainder of this paper might reasonably be seen as an extended footnote to this idea of "mathematical tool as expressive instrument." We will focus primarily on the use of HyperGami for the creation of polyhedral paper sculptures—what we have dubbed "orihedra" (suggesting a blend of "origami" and "polyhedra"). An example of this notion can be seen in Figure 1, depicting a family of polyhedral penguins created with the software. The individual elements of these figures—the heads, bodies, feet, and bowtie—are customized variants of particular classical shapes (the dodecahedron, cuboctahedron, pentagonal prism, and tetrahedron). Each of these shapes has moreover been decorated with (a relatively simple subset of) the tools provided by the software.

Figure 1. A family of three HyperGami polyhedral penguins.

The following (second) section of this paper presents a brief description of the HyperGami application and some of its major features; the third section illustrates a variety of specific techniques for polyhedral design in HyperGami by showing how the software may be employed to create the penguins in Figure 1. The fourth section describes our ongoing work with elementary and middle school students, and the types of sculptures that these students have made. Finally, the fifth section discusses some of the current strengths and limitations of the software; places this work in the context of related work in computational design, educational computing, and polyhedral modelling; and outlines several directions for ongoing and future research.

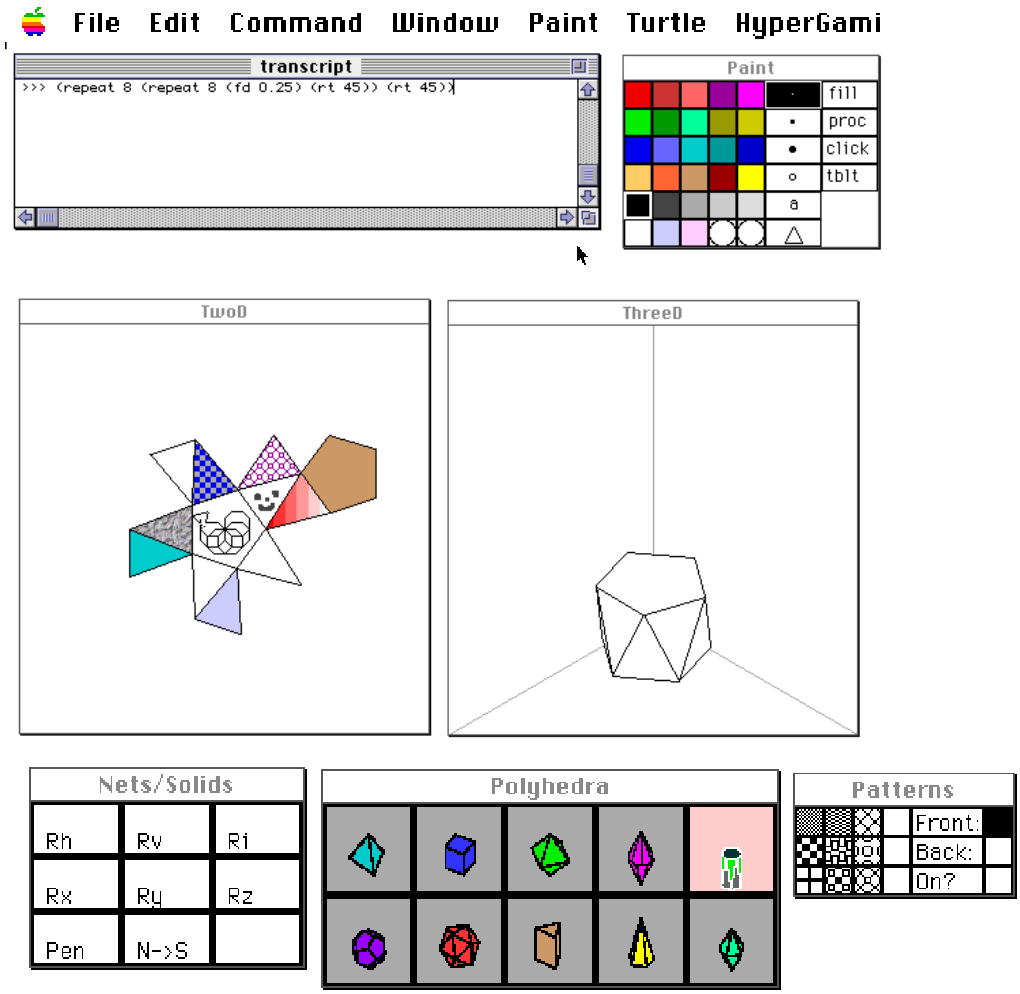

Figure 2. The HyperGami screen in the course of a sample project: the user is creating a decorated pentagonal antiprism. The various windows are described in the text (additional optional windows are not shown here). The three menu titles at right are specific to HyperGami; the four text titles at left are provided with the basic MacScheme environment. In the figure, the user has evaluated a turtle-graphics expression, and the turtle (positioned in the pentagon toward the center) is in the course of drawing a rotated-polygon pattern.

2. HyperGami: A Brief Overview of the System

HyperGami is written by the authors in the MacScheme dialect of Lisp [S2], and runs on all color Apple Macintosh computers with approximately 12-14M of memory. In its structure, HyperGami is a "programmable application" [Eisenberg, 1991], in that it includes both a variety of direct manipulation interface features as well as an extensive programming language that has been built "on top of" the basic MacScheme environment. The HyperGami language component thus includes all the standard Scheme primitive procedures and objects, augmented by a large vocabulary geared toward the creation of solids and decoration of folding nets.[3] Since the application-specific HyperGami vocabulary is built in Scheme, it also obeys, by and large, the default syntactic conventions of that language (i.e., very few HyperGami language constructs are "special forms" [Abelson and Sussman with Sussman, 1985] in Scheme).

The design of the system as a mixture of both linguistic and extra-linguistic elements is a principled one. On the one hand, the system's direct manipulation elements allow novice users to create relatively simple decorated shapes; on the other hand, HyperGami's interactive user-accessible language makes it possible for more advanced users to build their own personalized vocabularies of shape-creation and shape-decoration abstractions over time. We will briefly return to this issue in the fifth section of this paper; Eisenberg [1991] includes more extended argument along these lines, while Nardi [1993] provides eloquent arguments in support of the general idea of end-user programming.

To return to HyperGami: Figure 2 shows the system's interface in the course of a sample scenario. The standard pattern of activity for a HyperGami user begins with the creation or selection of a three-dimensional solid; the program automatically "unfolds" that solid into a two-dimensional folding net pattern, which may then be decorated, sent as output to a color printer, and folded into a paper model. In Figure 2, the conceptual core of the program is represented by the TwoD and ThreeD windows visible toward the center of the screen: the TwoD window is the one in which folding nets are displayed, and the ThreeD window shows the three-dimensional solid from which the matching folding net was derived. The Paint window includes a variety of direct manipulation tools for (e.g.) choosing pen colors and widths for the purpose of decorating folding nets; the Patterns window provides Macintosh-paint-program-style "patterns" to extend the range of coloring options; the Nets/Solids window includes controls for viewing solids from chosen vantage points; and the Polyhedra window provides an initial set of polyhedral shapes from which to choose.[4] (The specific functionality of these last several windows will be described as needed in this paper; for further detail see [Eisenberg and Nishioka, 1997].) Finally, the transcript window at the upper left of the screen

[3] HyperGami is available as free demonstration software, courtesy of J. Ulrich and LightShip Software; to obtain information about receiving HyperGami, send email to hg-support@cs.colorado.edu.

[4] The top row choices are (from left to right): tetrahedron, cube, octahedron, trapezohedra, and antiprisms. The bottom row choices are (from left to right): dodecahedron, icosahedron, prisms, pyramids, and bipyramids. The tetrahedron, cube, octahedron, dodecahedron, and icosahedron are the five regular (or "Platonic") solids [Holden, 1971]; the remaining five choices specify classes of solids. If, for instance, the user selects the icon for "prisms", she is presented with a dialog box in which to choose the number of sides in the (regular) base of the prism, as well as the height of the solid. Additional examples and detail on these matters will be provided later in this paper.

is the MacScheme interpreter. There are two more (optional) windows, not shown in Figure 2: these provide additional choices of starting polyhedra (the thirteen "Archimedean" solids and their duals [Holden, 1971]).

Figure 2 depicts a scenario in which the user has chosen to create a decorated pentagonal antiprism. The TwoD window reveals a folding net for the shape which the user is currently decorating. Three faces of the net have been filled with a solid color; two others have been filled with patterns; another with a granite-like texture; another with a color gradient; and another has been decorated "by hand", using the mouse as a pen. Toward the center of the figure, still another face of the folding net in the figure has been decorated using a turtle-graphics pattern—a "rotated polygon" figure of the kind described in [Abelson and diSessa, 1980]. In short, then, the decorative techniques available in HyperGami span a wide range, including techniques available through palette selection and Scheme language expressions. Beyond this brief description, however, this paper will not focus on the decorative techniques available in HyperGami (except as they arise in the course of future examples); rather, we will emphasize the elements of the program geared toward the creation of customized solids.

As a first step in pursuing the topic of solid construction, it is worth considering Figure 2 yet again. The example shows a decorated antiprism under construction; the process can be initiated by selecting the "antiprism" icon (top row, fifth from the left) in the Polyhedron palette. When this choice is made, the user is presented with a dialog box asking what sort of antiprism is desired (in particular, the height of the antiprism and the number of sides in the base, which is assumed to be a regular polygon—cf. [Holden, 1971], p. 61). Now, when the user selects the desired values, HyperGami generates both a *solid object*, and an associated *folding net object*. The former is a data object whose primary contents are the three-dimensional vertices, edges, and polygons of the given solid; the latter is a data object whose primary contents are the two-dimensional vertices, edges, and polygons of the net. In addition, the folding net object contains a set of "maps"—Scheme procedures—which, when called on a two-dimensional point in the net, will return the corresponding point on the surface of the three-dimensional solid; this is useful for certain advanced decorative techniques.[5]

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[5] Note, by the way, that the mapping between points in the net and points in the solid is one-to-one in the interior of the net, and many-to-one on its boundary. For example, several vertices at the boundary of the net may correspond (when folded) to one vertex of the eventual solid. Thus, there is no ambiguity in asking, for a given net point, for its corresponding solid point; but if we wished to employ this idea "in reverse", we might have to ask, for a given vertex of a solid (or a point on one of its edges), what the set of corresponding folding net points might be.

Importantly, in the Figure 2 example, the HyperGami folding net has been generated "on the fly" from its corresponding solid. Most of the starting set of polyhedra are in fact accompanied by "pre-stored" folding nets, but others (the prisms, antiprisms, pyramids, bipyramids, and trapezohedra) designate entire classes of solids. Thus, when one of these icons is selected (and additional specifying information is provided, like the antiprism parameters mentioned above), HyperGami first generates a brand-new solid object, and then employs a folding-net-generation algorithm to create a net for the solid. Figure 3 illustrates the algorithm in operation: the essential idea is that of a heuristic search in which the folding net is continually "expanded", growing step by step as new polygons are added. In each expansion step, a "frontier face" F is chosen on the boundary of the current folding net, and its corresponding face F' is found in the solid; the algorithm examines those solid faces adjacent to F' to see if any are as yet unaccounted-for in the growing net; and, if there is some as-yet-unaccounted-forsolid face G' adjacent to F', the algorithm attempts to place a new face G, congruent to G', at the appropriate spot on the boundary of the net. Note that this attempt may not be successful: for instance, the new face G, when placed so as to adjoin F in the growing folding net, may unfortunately overlap some earlier-placed polygon in the net. In this case, HyperGami's algorithm simply does not place the face G at this particular spot adjoining F, and the algorithm tries to place some alternative unaccounted-for polygon somewhere on the boundary of the net; ultimately (it is hoped), as the algorithm proceeds, the face corresponding to G' will eventually be placed at some plausible alternative location in the folding net where it will not overlap any other net polygons.

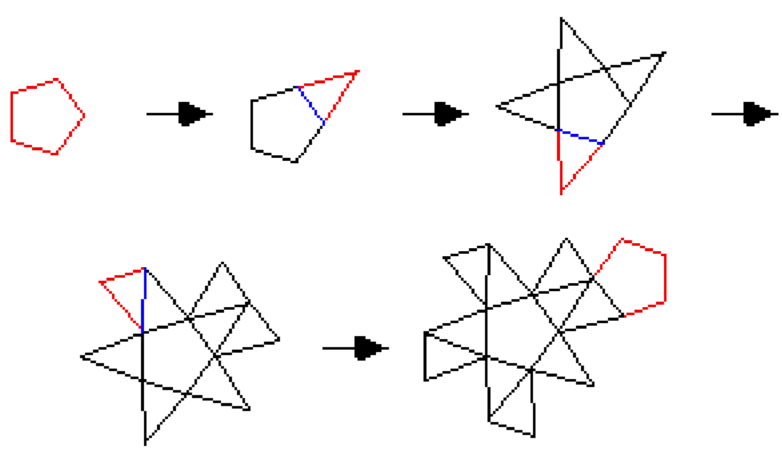

Figure 3. A series of "snapshots": steps in the generation of a folding net for the pentagonal antiprism. The program first places one of the pentagonal faces of the antiprism (top row, left); then places a neighboring triangle (top row, center); then additional triangles (top row, right; bottom row, left). The completed net is shown at right of the bottom row.

It is worth mentioning that this is only a skeletal outline of HyperGami's unfolding algorithm; and the problem that the algorithm is attempting to solve is quite tricky. In fact, the *general* question of whether a given solid is capable of being unfolded into a single non-overlapping two-dimensional pattern is (to our knowledge) an unsolved problem in computational geometry [Croft *et al.*, 1991]. It is easily shown that some solids *cannot* be unfolded into such a single net if the unfolding process is restricted, as in HyperGami, to "cutting" the solid only at its edges. Conversely, many solids are obviously capable of generating more than one alternative net (consider, e.g., the number of distinct possible unfoldings of a cube). In any event, HyperGami's algorithm is not guaranteed to succeed: if it produces a folding net for a solid, the net will be indeed be a correct one, but the program may fail to produce a net for a solid that in fact possesses one.[6] These technical considerations aside, HyperGami's unfolding algorithm has proven amply powerful in practice: it has thus far never failed to produce a folding net for a convex solid, and has successfully produced single-piece folding nets for numerous complicated nonconvex solids as well. In fact, we have recently extended this standard algorithm so that (with a bit of additional work) especially complex polyhedra may be unfolded into multiple pieces; this newly-extended algorithm is thus capable of producing a (possibly fragmented) net for any polyhedron whatsoever.

While a thorough description of HyperGami's two unfolding algorithms (both standard and extended versions) and their features is beyond the scope of this paper, the purpose of describing these ideas here is to emphasize the pervasiveness of the "unfolding problem" in the discussion of custom solid design that follows. Although most of HyperGami's starting polyhedra are stored with pre-computed folding nets, the brand-new custom solids created in the course of making orihedra must be "unfolded" by the program itself. We now turn to a specific example of the construction of an orihedron in which several custom shapes will be created.

3. How to Make a Penguin: Creating a Polyhedral Sculpture with HyperGami

In this section, we illustrate the types of HyperGami operations that can be used to create new solids by means of a sample project—namely, constructing the penguins shown in Figure 1. Note that our initial discussion of this project will focus only on the mathematical techniques needed to create new ingredients for sculptures—and not the design skills needed to conceive of those

[6] Much more detail about HyperGami's standard unfolding algorithm may be found in [Eisenberg 1996].

sculptures. We will return to the latter issue—the issue of how one might learn the craft of designing in paper sculpture—later in this paper.

3.1 The Penguin's Body: "Stretching" a Solid

In creating the penguin's body, we begin with the cuboctahedron—a semiregular solid (i.e., one whose faces are composed of more than one type of regular polygon—in this case, six squares and eight equilateral triangles). The cuboctahedron can be selected through HyperGami's palette of the thirteen Archimedean solids [Holden, 1971], shown in Figure 4.

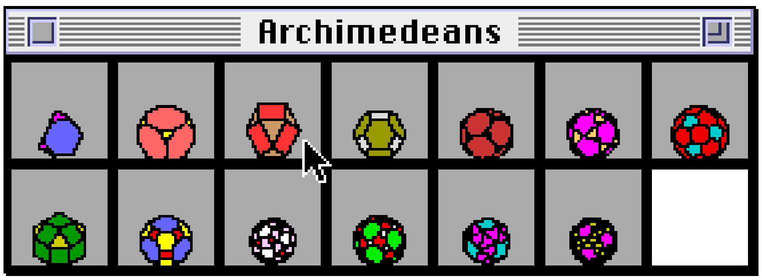

Figure 4. HyperGami's palette for the thirteen Archimedean solids. The user is in the process of selecting the cuboctahedron (top row, third from left).

Figure 5 shows the idea of how we would like to proceed in making the body of the penguin: at the left, we have the standard cuboctahedron, while at right we have a new variant of that solid, created by "stretching" the original shape along its vertical axis.

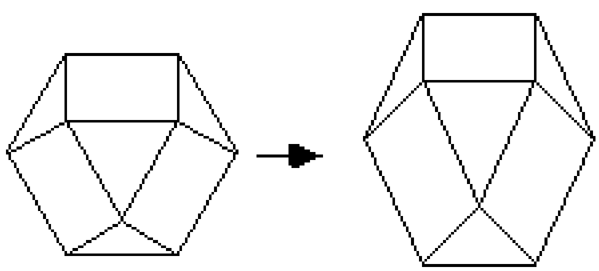

Figure 5. "Stretching" a cuboctahedron into a taller shape.

To perform this operation in HyperGami, we first select the cuboctahedron; then we select a menu choice that allows us to apply several types of linear maps (rotations and scales) to our already-chosen solid. The system presents us with the dialog box shown in Figure 6; and in this particular instance we stretch the cuboctahedron by 1.4 along the z-axis. Once this choice is made, the system now creates a "stretched" version of the cuboctahedron, and computes a two-dimensional folding net for the solid, as shown in Figure 7.

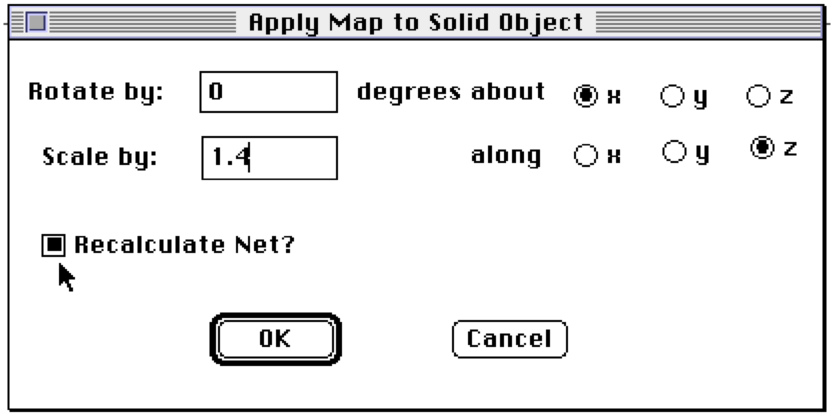

Figure 6. A dialog box prompts the user for linear map parameters with which to rotate and/or scale the current solid object. The user has also chosen to have the system calculate the folding net for the newly-mapped solid by selecting the box toward the bottom left.

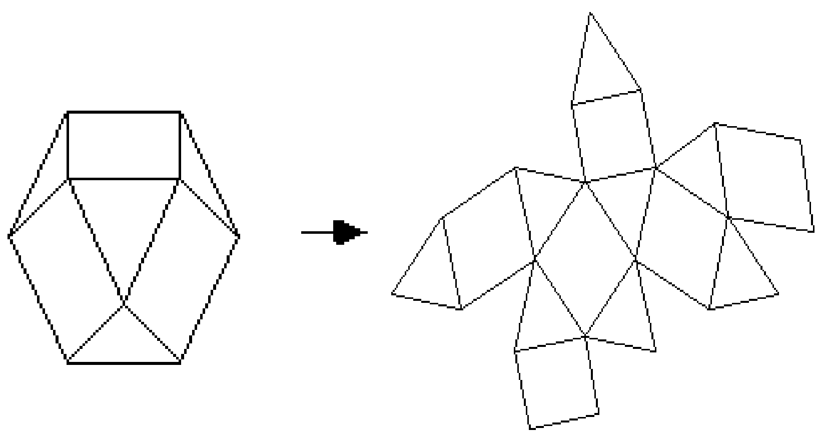

Figure 7. The stretched cuboctahedron is unfolded by HyperGami to produce a folding net.

Once the new folding net for the "stretched" cuboctahedron has been generated as in Figure 7, it may be decorated in HyperGami to produce a suitable penguin body. (For the Figure 1 sculptures, two of the triangles were decorated in white—to suggest the penguin's chest and belly—and the rest were decorated in black.)

3.2 The Penguin's Head: "Capping" the Face of a Solid

The head of the penguin in Figure 1 is a variant of the dodecahedron, created by "capping" one face of that solid. The idea is suggested by Figure 8: we take the original dodecahedron and make a new solid by adding a single vertex *v* outside one of the faces of that dodecahedron, drawing new edges to *v* . The result is a solid that looks rather like a head with a beak.

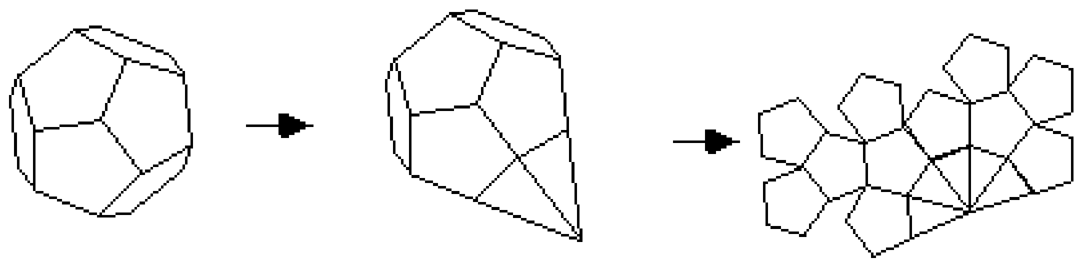

Figure 8. A dodecahedron (left) is "capped" to form a new shape which is then "unfolded" (at right).

In this case, the technique for generating a capped solid would be performed using HyperGami's Scheme language operations: a function named add-cap-vertex is used to generate a new solid from an older one. Here is the expression used to generate the shape at the center of Figure 8:

```
(add-cap-vertex
  *current-solid-object* (read-a-current-solid-face) 0.7)
```

The `add-cap-vertex` function takes three arguments. The first argument is a solid (in this case, our "current solid"—the dodecahedron—which has the default name `*current-solid-object*` indicating that it was the last solid object selected); the second argument is a specific face of that solid (here, the `read-a-current-solid-face` procedure allows us to select that face with the mouse); and a "cap height" indicating how far the new vertex should be from the center of the capped face. When this expression is evaluated in HyperGami's language interpreter, the system prompts us to select a face of the dodecahedron, and returns the solid shown at right in Figure 8. This new form may now be unfolded; and the folding net may be decorated to suggest a penguin's head. (In the case of the Figure 1 sculptures, the beak was done in orange, the face pentagons in white—with hand-drawn eyes—and the rest of the head in black.)

3.3 The Penguin's Feet: Creating Custom Prisms

The feet of the penguins in Figure 1 are designed by creating customized prisms. HyperGami allows the user to generate a prism based upon an arbitrary polygon—whether that polygon is hand-drawn, created by specifying a sequence of points through a dialog box, or created by language operations. The pentagon used for the penguin's feet was in fact created by a Scheme expression:

```
(define penguin-foot-polygon
  (make-closed-polygon
    (make-point -0.5 0)(make-point 0.5 0) (make-point 1.1 1.86)
    (make-point 0 2.5) (make-point -1.1 1.86)))
```

This polygon was "turned into" a prism by the following expression:

```
(make-general-prism-solid-object penguin-foot-polygon 0.5)
```

The `make-general-prism-solid-object` function takes two arguments: a base polygon, and a prism height. The result returned by this expression is the solid shown in Figure 9, which may now be unfolded and decorated in the usual way. (The feet of the Figure 1 sculptures were decorated in orange.)

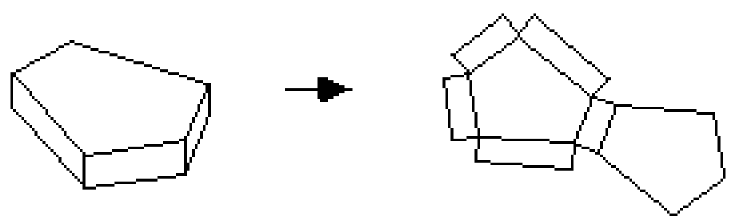

Figure 9. A customized pentagonal prism is unfolded into a two-dimensional net.

3.4 The Penguin's Bow Tie

The bow tie of the large penguin in Figure 1 is simply composed of two tetrahedra; the tetrahedron is in fact one of the five regular solids and may be selected (along with its folding net) directly from HyperGami's **Polyhedra** palette. The tie of the Figure 1 sculpture was decorated in red.

3.5 Additional Solid-Customization Operations: an Algebra of Solids

In the course of creating the penguin sculpture of Figure 1, several important techniques for creating new polyhedra were employed: adding a "vertex cap" over a face, "stretching" a solid (more generally, applying a linear map to the solid), and creating a custom prism from a given two-dimensional polygon. These are just several of the techniques available in HyperGami for creating solid objects; the program currently includes a large and growing repertoire of methods for making solid shapes.

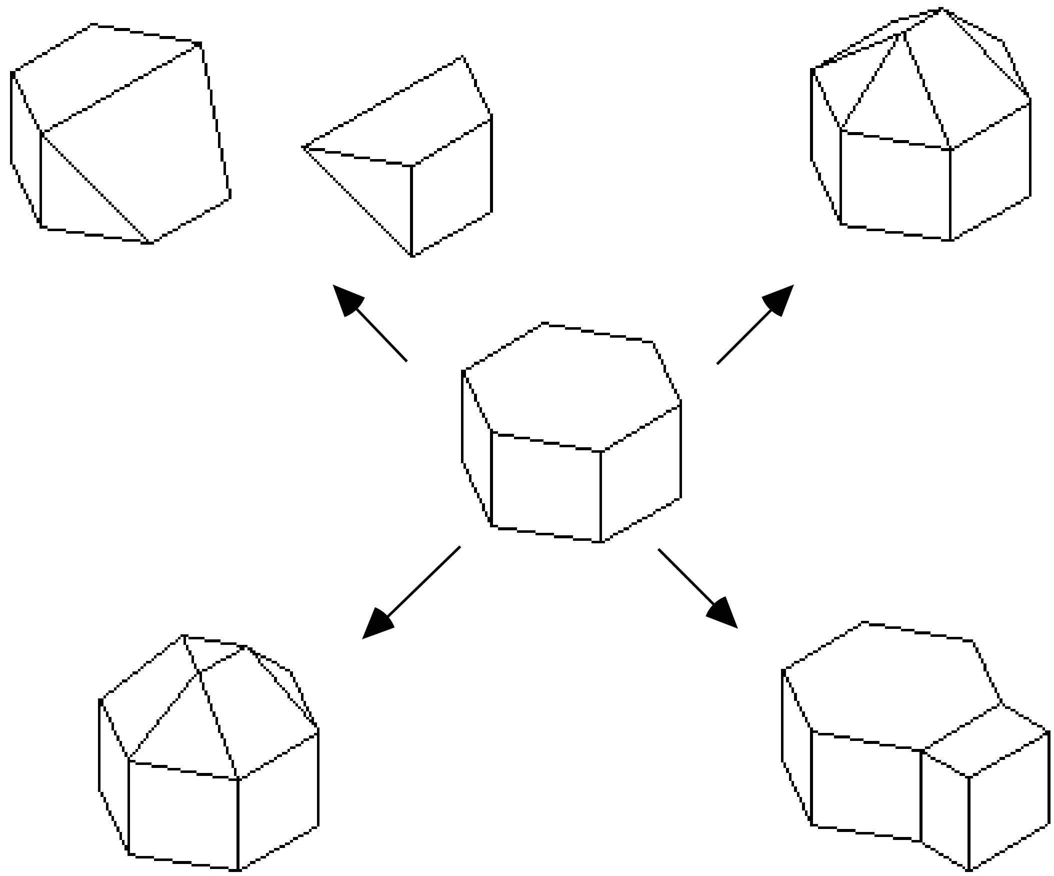

Figure 10. A hexagonal prism (at center) is sliced into two portions (upper left); extended with an "edge cap" (upper right); extended with a "triangle cap" (bottom left); or extended with a "prism cap" (bottom right).

A selection of these methods may be seen in Figures 10 and 11. Figure 10 shows a variety of customization operations applied to a sample shape—in this case, a hexagonal prism. Toward the upper left, the prism has been "sliced" into two pieces through a plane determined by three points; toward the upper right, the top hexagonal face of the prism has been "capped" with an edge (distinct from the "vertex capping" method used to make the penguin's head); toward the bottom left, that same hexagonal face has been "capped" with a triangle; and toward the bottom right, a rectangular face has been extruded from the shape to make what might be called a "prism cap". Figure 11 depicts several more operations as applied to a pentagonal antiprism. Toward the upper left, two triangles have "swapped vertices" to produce a new solid; toward the upper right, a "notch" has been placed in one of the upper horizontal edges of the shape; toward the bottom left, a vertex has been truncated from the shape (this could be seen as a special case of the slicing technique at the upper left of Figure 10); and toward the bottom right, the antiprism has been "joined" to another solid, the dodecahedron, to produce a composite shape.

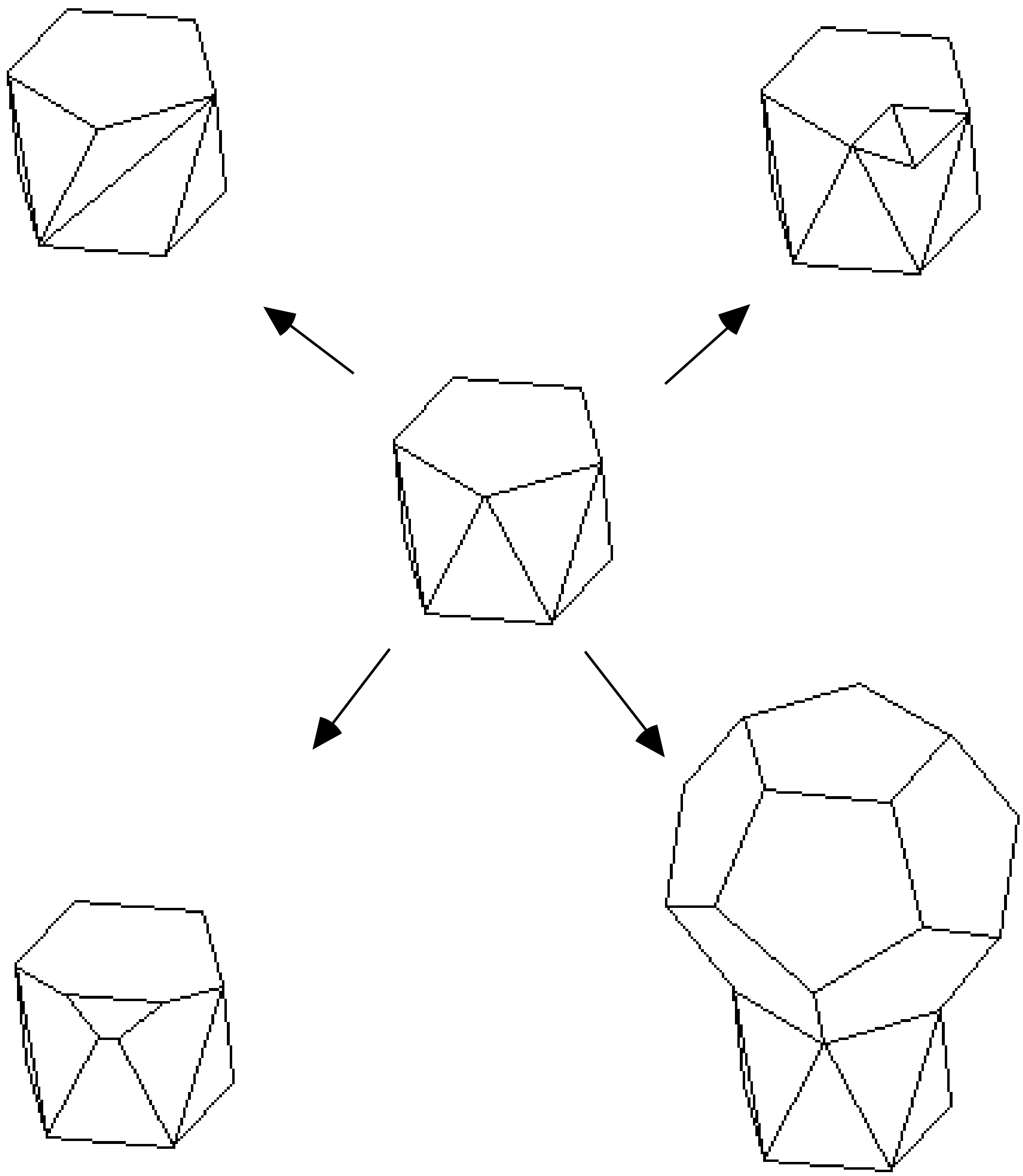

Figure 11. A pentagonal antiprism (center) is transformed by the "exchange of vertices" of two triangles (upper left); a "notch" is placed in an edge (upper right); a vertex is truncated (bottom left); and the solid is joined to a dodecahedron (bottom right).

Typically, these operations are invoked by using HyperGami's extended Scheme language; and they are written as functions that operate on solids to return new solids. For instance, the "triangle-vertex-swapping" operation in Figure 11 could be used by employing HyperGami's `exchange-triangles` function as follows (here, we have given the names `*antiprism-1*` to the antiprism solid object, and `*face-1*` and `*face-2*` to the two triangular faces whose vertices are involved in the transformation):

```
(define *new-shape*
  (exchange-triangles *antiprism-1* *face-1* *face-2*))
```

The result returned by the `exchange-triangles` function call is a new solid that is given the name `*new-shape*`. This shape, in turn, could be employed as the starting point for yet another customization. For instance, if `*face-3*` is the name of the upper pentagonal face of our new shape, we could write the following:

```
(define *new-shape-2*
  (add-cap-vertex *new-shape* *face-3* 1.2))
```

This expression returns a "capped" variant of `*new-shape*`, as illustrated in Figure 12. Overall, then, HyperGami's solid-customization functions may be thought of as constituting an algebra of solids: a collection of operations may be applied to solid objects to produce new solid objects, which may be operated upon yet again, and so on. As a result, HyperGami is capable of producing an endless variety of unusual or attractive solid shapes. Beyond these purely creative considerations, though, thinking in terms of an algebra of solids leads to productive mathematical ideas. Often, unexpected or unnoticed conceptual links between polyhedra spring from concatenating simple alterations to starting shapes. Figure 13 depicts several examples along these lines. Here, a pentagonal antiprism is capped at its top and bottom to produce an icosahedron; a cube is sliced four successive times through planes defined by sets of three vertices to produce a tetrahedron; an octahedron is truncated at every vertex, each edge sliced through its midpoint, to produce a cuboctahedron. In each of these examples, a connection between solids is represented in procedural terms—i.e., as a series of operations to produce one polyhedron from another. Moreover, in some cases, the very expression of this procedural connection highlights important mathematical relations between the two solids in question: for instance, when the cuboctahedron is produced from the octahedron via truncation, the fact that the truncation operation at each octahedron vertex is identical suggests that the cuboctahedron will have exactly the same symmetry group as its parent solid.

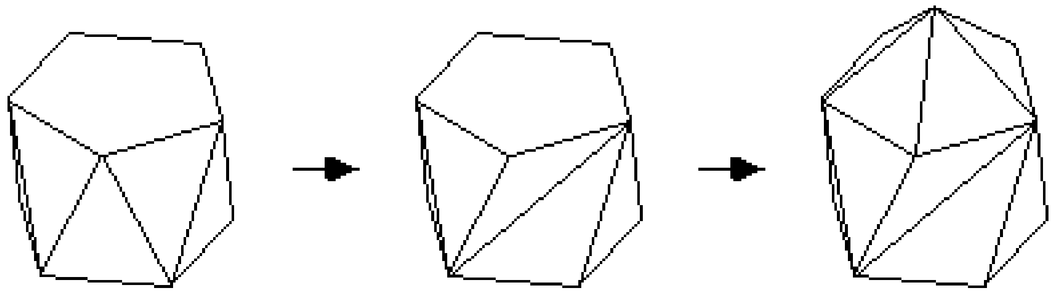

Figure 12. A pentagonal antiprism (left) has two triangles exchanged (center); this new solid is then capped at a face to produce still another shape (right).

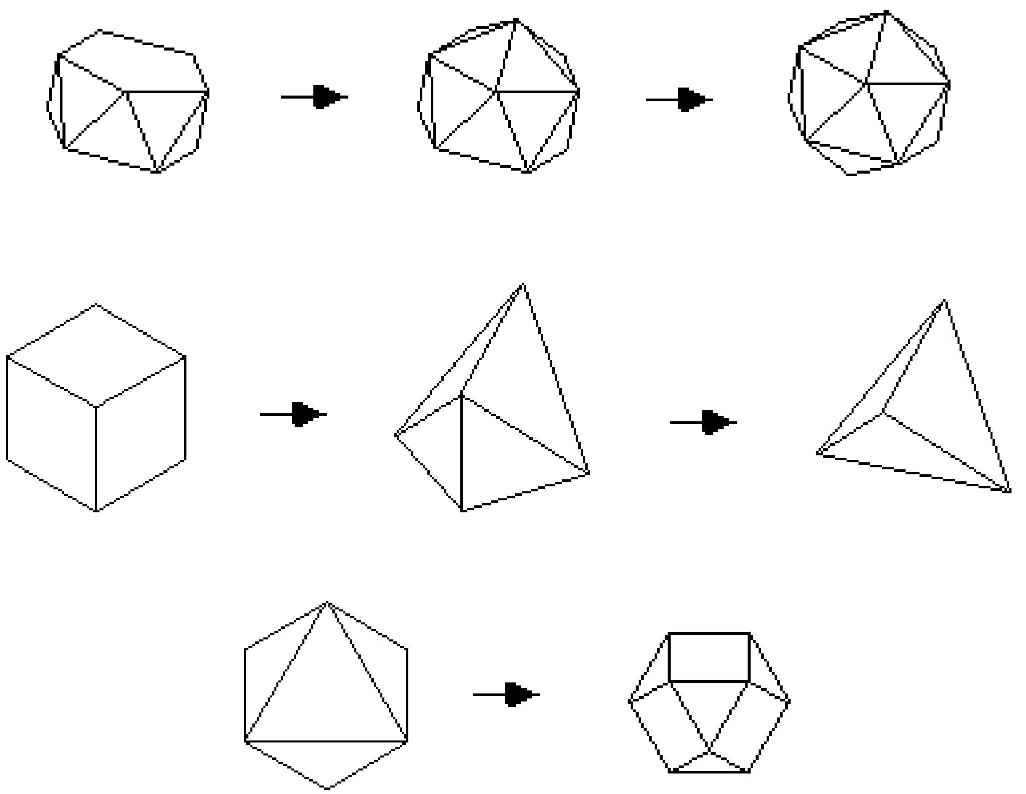

Figure 13. Several examples in which one polyhedron is linked procedurally to another through HyperGami's solid-customization operations. Top row: a pentagonal antiprism is capped at each pentagonal face to produce an icosahedron. Center row: a cube is "sliced" twice to produce the center shape, then twice more to produce a tetrahedron. Bottom row: an octahedron has each vertex truncated so that each edge is divided in half, resulting in a cuboctahedron.

In concluding this topic, it should be prominently noted that many of HyperGami's solid-customization operations have been only recently included within the system; and currently only a few are accompanied by direct manipulation tools (such as the linear-map dialog box shown earlier). Thus, with a few exceptions, HyperGami's solid-customization operations must be invoked by writing Scheme expressions; and while these expressions are typically brief (as in the examples shown earlier), they represent a more advanced use of the system than has yet been attained by our elementary- and middle-school students. We will return to these topics later, both in describing our experiences with students and in outlining plans for future work.

3.6 Orihedron Design as an Adult Mathematical Craft: Some Illustrative Examples

As mentioned earlier, our "explanation" of how to construct the penguin sculpture of Figure 1 focused on the specific mathematical techniques of creating customized polyhedra; as such, it ignored the (at least equally interesting) question of how to design a paper sculpture in the first place. We are still—we hope, forever—in the process of answering this question for ourselves, and thus can hardly claim to any expertise in the matter. Many of our own sculptures derive from observing forms in the real world, and then attempting to "analyze" those shapes into polyhedral forms. (Similar ideas for sculpting and three-dimensional design may be found in [Pearce and Pearce, 1980].) Often—as in the penguin sculptures—we have found that combinations of relatively simple shapes are able to suggest interesting subjects: the mushrooms of Figure 14 and fish of Figure 15 are made, respectively, from two and three pieces. Or, as in the case of the caterpillar in Figure 16, a few simple shapes have been employed repeatedly.[7]

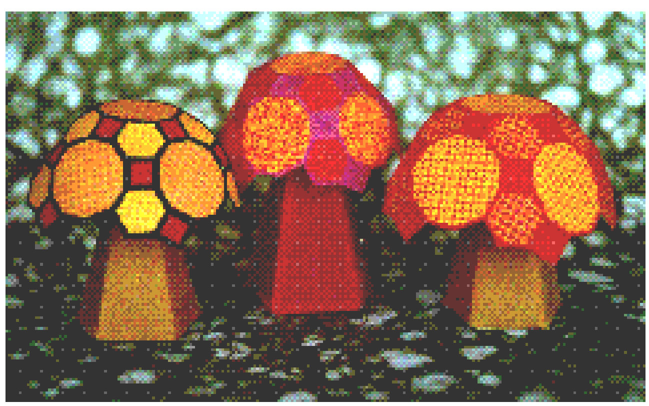

Figure 14. Polyhedral mushrooms (from the book "AlphaBetaHedra").

[7] The mushrooms are composed of a truncated pyramid (base), and half of a great rhombicosidodecahedron (cap). The fish are composed of a truncated trapezohedron (body) and two simple prisms (fins). The caterpillar is composed of capped hexagonal prisms (the segments of the body) and prisms built on a language-specified octagonal base (the sneakers); the eyes are tetrahedra.

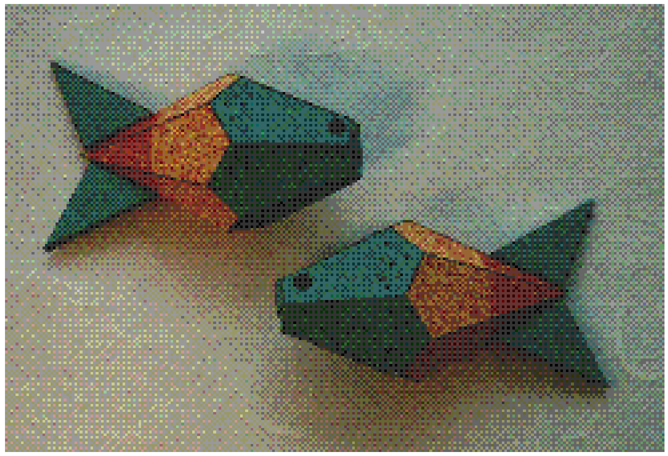

Figure 15. Polyhedral fish (from "AlphaBetaHedra").

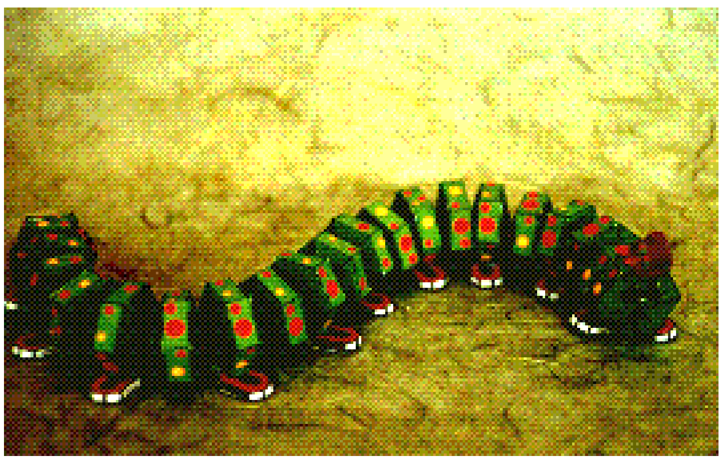

Figure 16. A polyhedral caterpillar (from "AlphaBetaHedra").

In creating more complex sculptures, we have sometimes found it helpful to examine three-dimensional models, and to analyze these (often abstract) forms into constituent polyhedra. The hippopotamus in Figure 17 is a case in point: this figure was created by approximating the forms in a stuffed toy version of a hippo. Quite probably, the already abstract nature of this toy was an important aid in finding the appropriate polyhedral forms; at any rate, we had had far less success studying photographs of the actual animal. In our figure, the head of the hippo—the portion which was most difficult to design—is made of two shapes, both based on the small rhombicosidodecahedron, one of the Archimedean solids: the rear portion of the head is in fact a slightly "squashed" version of the front portion (that is, the rear portion is constructed by applying a linear map that contracts the front-portion shape along one axis).

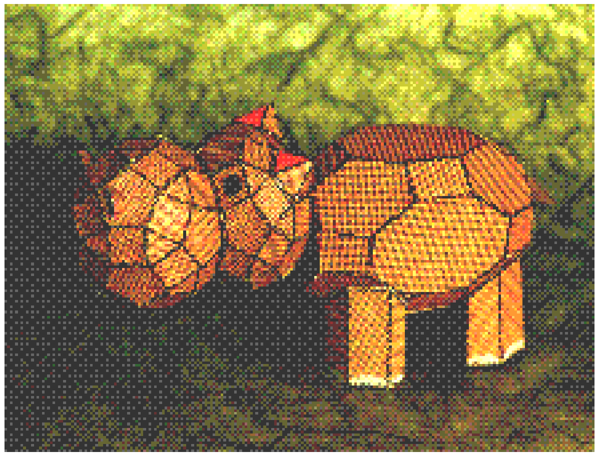

Figure 17. A polyhedral hippo (from "AlphaBetaHedra").

4. Experiences with HyperGami Students

Over the past two years, we have employed HyperGami in pilot tests with a total of 16 elementary and middle school children ranging in age from 8 to 13. Several undergraduates have also employed the system; but this paper will focus on our experiences with schoolchildren.[8] These students—who volunteered to experiment with the software—worked with us as individuals or in pairs, coming to weekly or biweekly sessions that lasted one to three hours; typically, students worked with us for approximately one semester. Not all the children's projects involved creating polyhedral paper sculptures with HyperGami; some projects focused on origami figures (the software may also be used for this purpose [Eisenberg and Nishioka, 94]). All of the children did choose, at some point, to create polyhedral figures (though not all of these were full-fledged "sculptures" representing nonmathematical objects like the penguins in Figure 1).

[8] It is also perhaps worth noting that HyperGami has been made available, free of charge, since earlier this year. During this period, the software has been distributed to over eighty individuals in response to their requests. We have little information to date, however, about how this distributed software has been employed.

Figure 18. A student-designed (and -constructed) paper sculpture of a brontosaurus.

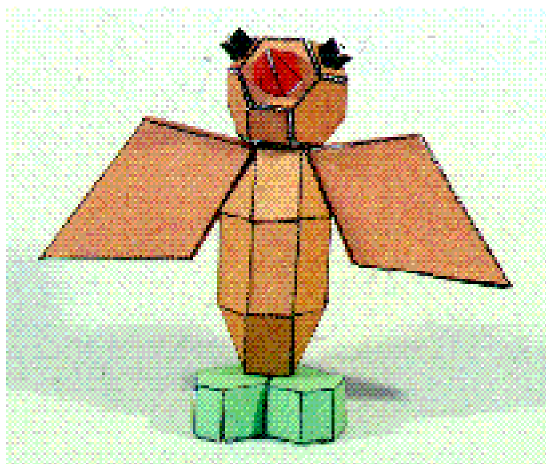

Figure 19. A student-designed construction, "SuperDuck".

4.1 Student-Constructed Orihedra: a Summary of Experiences

Among our 16 students, 12 employed HyperGami to create original orihedra. Many of these projects employed sophisticated geometric components. One 11-year-old girl used antiprisms and a truncated icosahedron to create a paper hippopotamus; a 10-year-old boy experimented with an elongated icosahedron to create an "alien head"; a 13-year-old boy expressed his fondness for dinosaurs by building a polyhedral sculpture of a brontosaurus (see Figure 18) that made use of antiprisms and a "stretched dodecahedron"; and a fifth-grade boy created “SuperDuck" (a super-hero duck) shown in Figure 19.

Yet another student construction—the orihedral circus clown shown in Figure 20—is of interest in that it highlights some important issues in the use of a program such as HyperGami by elementary school students. The head of the clown—a great rhombicosidodecahedron (one of the Archimedean solids)—was decorated by the student, but he did not quite have the dexterity to fold this shape on his own. Thus one of us helped him build the head; the other portions of the sculpture were created by the student on his own.

Figure 20. A student's HyperGami circus clown.

This anecdote is illustrative of a general pattern in the use of HyperGami—namely, that it calls upon skills of hand-eye coordination beyond those involved simply in using the software. Speaking broadly, the capabilities of our older students in this regard—their levels both of fine motor control and patience—have been very noticeably greater than those of the younger students. Another anecdote may help to elaborate on this observation: at one point this spring, we held a small "folding party" for the students working with us at the time (as well as a few guests). The project during this party was to build a "paper crystal" composed of a large number of one particular shape that happens to tile space—the truncated octahedron. The plan was to first design a collection of these shapes and then glue the individual shapes together. (Compare [Williams 1979], p. 167.) We soon observed that, in practice, the success of such a project required a level of precision in construction that few of our younger students in fact possessed; the difference between those crystal elements made by adults (and older students) and those made by younger students was visible to the casual observer.

Unquestionably, younger children are not capable of designing or constructing sculptures as sophisticated as those done by adults. Still, there are many different ways in which they may interact with HyperGami according to their own interests and abilities; and there are different pathways by which they can be introduced to the creation of polyhedral sculptures. In some cases, children have begun with our basic designs and used the software to create their own custom decorations. While working with our "pre-stored" polyhedra, the children can concentrate on learning to use the decorative features of the system without the added complexity of manipulating the solid. This serves as a gentle introduction to the software in which the child arrives at a satisfying finished product. For example, during her earliest interactions with the system, an 8-year-old who worked with us made penguins (of which she was considerably proud) by filling the outlines of our penguin nets with her own designs.

Another stepping-stone in interaction with the system is for students to create individual custom polyhedra before attempting to design a multipart sculpture. Our students have spent sessions applying linear maps to the built-in Platonic and Archimedean solids to stretch them out along different axes; they have experimented with the number of sides and heights of pyramids; and they have employed functions to truncate shapes at vertices. When the focus is primarily upon creating a single custom polyhedron, the process from design to construction can often be completed in a session or two, so students emerge with tangible results of their work after a relatively brief period.

Although the focus of our HyperGami sessions is not on teaching specific programming skills, 10 of the students have worked with us to employ various Scheme programming techniques to further customize their sculptures. A number of students have made their own "custom color objects" in Scheme by experimenting with changing parameter values for red, green, and blue. A 13-year-old boy also made more intricate use of color by extending a pre-existing procedure to fill a region with speckles. The procedure used the idea of random number generation to select the color of a pixel at a given point, and the student generalized the sample procedure that selected from five colors to one that selected from twenty.

In addition to experimenting with colors through programming, students have also been introduced to other decorative techniques. One of our activities in a previous semester was to have each student decorate a body segment and a sneaker of a "caterpillarhedron" similar to the one shown in Figure 16 . A 13-year-old girl worked with us to decorate her contributed parts with turtle graphics designs. Other students have similarly had an introduction to turtle graphics routines through placing patterns of repeating geometric shapes on the faces of cubes, dodecahedra, and cuboctahedra.

Students have also employed programming to create specialized polyhedral shapes. They have worked with custom prisms in the same way that the penguin's feet were created in an earlier section of this paper. The 10-year-old creator of "SuperDuck" worked with us to design custom prisms for both the wings and feet of the duck. The 13-year-old boy who created the brontosaurus helped to design custom prisms for its feet, and then later participated in designing a different pair of custom prisms as wings for a second dinosaur.

In designing their own custom polyhedra, students have the opportunity to rethink, and personalize, solid geometry. One student who had worked with the software began by describing "basic shapes... like cubes, like three-dimensional triangles" and then continued, "I think everything builds off of these shapes. They are sort of like the primary colors, you know, you mix to get different colors from the three primary colors and you get different shapes with the solid shapes...." Another student, pursuing a biological analogy, stated that "there are many, totally many kinds of polyhedra out there, but they all start from a simple shape, which is basically their 'genes'...."

Moreover, students often look upon their finished creations with both affection and pride. During one of her early sessions with the software, a 10-year-old girl worked diligently to decorate and construct a dodecahedron. She was so fond of the resulting shape that she named it "Fred" and took it to school with her. A 13-year-old girl created a stellated cuboctahedron and later reported that it had been on display on her mantle at home. The 13-year old who made the brontosaurus was by nature quite shy, but took the initiative to show his sculpture to one of his teachers.

The students have also made solid geometry into a more personal experience by turning their polyhedral shapes into gifts: a pair of 10-year-old girls made holiday tree ornaments as surprises for their parents, an 8-year-old girl created a polyhedral boat as a present for her mother, and a 13-year-old boy employed HyperGami's paint tools to write the name of his math teacher on the faces of a gift icosahedron. For these students, then, HyperGami served the type of role that we described at the outset of this paper—namely, as a means of rendering mathematical activity less "austere" and more expressive.

4.2 Orihedra as a Vehicle for Mathematical Ideas

Our experiences to date have given us reason to believe that orihedron-building is a promising vehicle for introducing solid geometry to children: as the previous section documented, our students have been able to design and create personalized sculptures, and they have often exhibited a striking level of pride and affection for their creations. It is fair to ask, in this context, what the precise mathematical content (or benefit) of working with orihedra might be. Is orihedra construction a truly mathematical enterprise, and if so, in what way?

Our answers to these questions are admittedly tentative and, at least in part, aesthetic. Certainly, we can point to traditional sources in arguing for the value of mathematical papercrafts and polyhedral modelling generally. There is a long and venerable tradition of mathematical writing describing the joys of polyhedron-construction, including excellent books by Hilton and Pedersen [1994], Wenninger [1971], Cundy and Rollett [1951], and—our personal favorite—an inspiring book by Holden [1971]. A set of eighteenth-century cardboard polyhedra may be seen on display at the University of Göttingen [Mühlhausen, 1993]; while Banchoff [1990] describes the pioneering work of the late nineteenth-century educator Friedrich Froebel in terms that echo the concerns of our own work:

> "Froebel... devised a set of 'gifts' to introduce children to notions of geometry in several different dimensions. His philosophy was clear: if children could be stimulated to observe geometric objects from the earliest stage of their education, these ideas would come back to them again and again during the course of their schooling, deepening with each new level of sophistication.... Froebel and his colleagues created geometrical gifts from materials available to them, primarily wood, paper, and clay." [p. 11-14]

Banchoff adds,

> "In the present-day rush to prepare students for calculus before they go off to college, we are systematically shortchanging them by ignoring the most practical and useful of all geometry—the geometry of our own dimension.... If we wait until students have developed a great deal of arithmetic sophistication (and a great many misconceptions) before we encourage them to think about solid objects and the interaction between different dimensions, we may be depriving them of the chance to appreciate the full power and scope of geometry." [pp. 13-14]

Senechal and Fleck [1988] likewise offer a spirited argument in favor of just the sort of activities that HyperGami promotes:

> "Our view... is that in a restructured geometry curriculum a central role should be given to the study of three-dimensional forms. We can envision, for example, a high-school course on the principles of three-dimensional structure. It would begin with a survey of significant forms, both from nature and those of human design; the polyhedral shapes such as pods and crystals, molecular and crystal structures, and trusses and other bridge constructions. The next step would be to build some of these forms, focusing on basic structural units and how they fit together. This would lead to important questions about the geometric principles on which these forms are based. Students would be led gradually to systematize and formalize their geometric knowledge.... We also believe that because spatial geometry is one of the small group of subjects that truly comprise 'the basics', it should be integrated with instruction in art, biology, chemistry, and physics." [pp. 264-5]

The concerns of these educators are heard as well from more "official" sources. In its most recent set of standards for mathematics education in grades 5-8, the National Council of Teachers of Mathematics [1989] prominently recommends that students should be able to "visualize and represent geometric figures with special attention to developing spatial sense" [p.112]; the Council's recommendations for high school mathematics include the suggestion that students should be able to "interpret and draw three-dimensional objects."[p. 157]

There is some evidence in the literature of mathematics education that the construction of paper polyhedra has beneficial effects on the performance of related geometric tasks. In Piaget and Inhelder's *The Child's Conception of Space* [1948], the authors relate experiments in which elementary-school-aged children were asked to draw unfolded representations of simple shapes (including a cube and pyramid). Looking at the performance of their subjects, Piaget and Inhelder report that "the child who is familiar with folding and unfolding paper shapes through his work at school is two or three years in advance of children who lack this experience" (pp. 275-276). Brinkmann [1966] describes a "programed instruction" curriculum for spatial visualization, lasting approximately 3 weeks, which included among other elements cutout patterns for geometric solids (as well as an "object kit" of various pre-constructed solids). Brinkmann's experimental group achieved significantly better gains than a control group on a standardized test of space relations. In one particularly interesting study—though not directly related to construction of paper polyhedra—Mitchelmore [1980] found a difference of about 3 years of development between British and American schoolchildren of comparable ages in tasks involving three-dimensional drawing (the British students were more advanced). He writes that the "size of the Bristol-Columbus difference... is a truly amazing result.... A more likely explanation for the observed difference lies in the school mathematics curriculum. In the author's experience... English teachers tend to have a more informal approach to geometry, *to use more manipulative materials* in teaching arithmetic at the elementary level, and to use diagrams more freely...." [p. 212; emphasis added] In the same discussion, Mitchelmore also speculates that "[t]he mere handling of 3D shapes in a structured learning situation may promote growth in spatial ability" and that British children may employ more constructional toys than American children.

It is our position that the design and creation of orihedra offers students a meaningful opportunity to engage in the exploration of three-dimensional shapes recommended by writers such as Senechal and Fleck. We also believe that building orihedra is at least consistent with the experiences that Piaget and Inhelder found to be beneficial for the population of children that they studied. In our very most recent group of students (not counted among the 16 described earlier in this section) we have begun to look for improvement over time in spatial tasks related to the understanding of solids and their folding nets. While our sample of students is small (and self-selected), and our data still incomplete and preliminary, there are at least some positive indications of improvement in certain spatial exercises as a result of working with HyperGami. Figure 21 shows an example of two folding nets—for the cube and octahedron—drawn by a 9-year-old student both before and after eight sessions working with HyperGami (a total of 9 hours 45 minutes, spanning sessions over fewer than three months). The student's representation of the cube net has progressed in this time from an imprecise (and incorrect) shape to a correct net; the representation of the octahedron net is still not correct in the "after" version (it includes only seven triangles), but is much more systematic than the spiky shape drawn before building polyhedra. Again, it should be noted that this example is a single data point; and of course the student's improvement may well have occurred through any number of interventions (i.e., HyperGami may not have been the most efficient means of effecting this change for the student). Nonetheless, the change in the student's representations is noticeable, and plausibly influenced by his work in building paper solids with HyperGami.

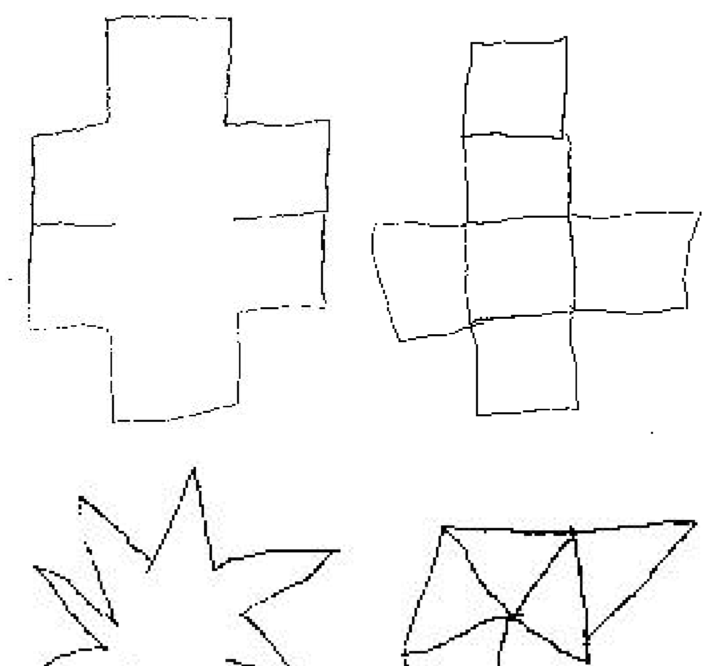

Figure 21. Top row: A nine-year-old's representation of an unfolded cube, before (left) and after (right) working with HyperGami. Bottom row: The same student's representation of the octahedron net before (left) and after (right) working with HyperGami.

5. Ongoing and Related Work; Future Directions

Thus far, we have described the HyperGami software; how it may be used to create polyhedral sculptures; and the experiences both of ourselves and of our students in building these sculptures. In this concluding section, we enumerate some of the current limitations of HyperGami as a sculpture-building medium; we place HyperGami in the context of other, related efforts in polyhedral modelling and educational computing; we outline several of our current research directions; and we present our views on those important directions for future research that are suggested by our experiences with HyperGami to date.

5.1 The Current State of HyperGami

HyperGami, while a readily usable program, may still be fairly characterized as a work-in-progress, and a system that could well be improved along a variety of dimensions. The program lacks many features that could enhance its role as a medium for the design of paper sculpture:

- In HyperGami, only one polyhedron may be designated the "current solid object"; this means in turn that the system is geared toward presenting only one solid at a time on the screen. Thus, when the user is designing a multipart sculpture, such as the penguins of Figure 1, she is unable to view how the several parts of the sculpture will look in combination: it is, to all intents and purposes, impossible in HyperGami to see the entire penguin figure on the screen. The most immediate problem that arises in this context involves the comparative scales of the printed-out shapes: we have often found, in the construction of an orihedral sculpture, that we have needed to print out several drafts of a particular shape at several alternative sizes in order to match that new shape to a partially-completed sculpture. Extending HyperGami so that it can present multipart sculptures all at once on the screen would alleviate this difficulty in matching the scales of distinct polyhedral components.

- HyperGami's folding nets do not include the extra "tabs" typically used by craftspeople who work in paper. In practice, this is not a major issue: generally it is easy for the user to add tabs "informally" while cutting the folding net out of paper. Still, it might be helpful for the system to add tabs to the folding nets automatically [Hayes, 1995].

• In HyperGami, the folding net, and not the solid object, is the target of decoration: that is, the user decorates the two-dimensional folding net on the screen. HyperGami does include a feature through which the decoration on the folding net may be "transferred" to the three-dimensional solid view; this feature is described in [Eisenberg and Nishioka 1997], and illustrated in Figure 22. This feature is extremely useful in permitting users to see what a decorated three-dimensional solid will look like before it is actually printed and folded; nonetheless, in practice it often seems to us that it would be desirable to begin by decorating the three-dimensional solid as opposed to the folding net. A worthwhile addition to HyperGami, then, would be a facility allowing users to decorate the solid surface on the screen and to transfer those decorations to the folding net for printing.

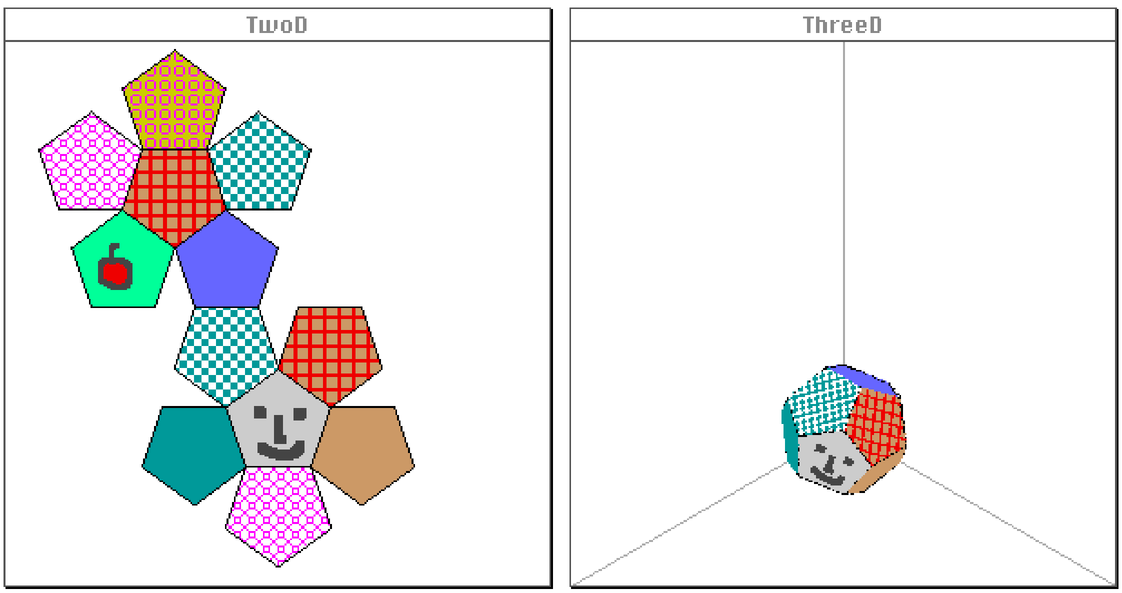

Figure 22. The decoration on the folding net of the dodecahedron (at left) is transferred to the three-dimensional solid view (at right).

While these additions would certainly be improvements to HyperGami, our most immediate priority in developing the program is to make a variety of incremental improvements to the interface and documentation. Much of our work on the interface will center on the broad task of allowing operations on solids to be performed more easily, with a mixture of programming and interface operations. One likely enhancement will be a palette of iconically-represented solid operations (e.g., one selection represents "capping a face", another represents "truncating a vertex", and so forth): when one of these operations is selected, an editable sample HyperGami expression will be presented illustrating how the given operation might be used. Thus, selecting the "truncating a vertex" icon would cause the system to present the following editable operation:[9]

```
(truncate-solid-at-vertex
   *current-solid-object* (read-a-current-solid-vertex) 0.5)
```

Perhaps a more interesting area of development in HyperGami is our current effort to create "papercrafts-based classroom activities". Our intent in this effort is to forge more explicit links between principles of solid geometry and orihedron-building, and to engage students in reflection on the geometric properties of the solids that they use in their sculptures. In effect, our goal is to strengthen the mathematical content of orihedron-building that was only alluded to in the previous section. One might (for instance) prompt some geometric reflection on the shape used for the body of the penguin sculpture in Figure 1: what symmetry operations of the (original) cuboctahedron are retained by the "stretching" operation used to produce the new shape? What symmetry operations are retained once the solid has been decorated? Similarly, one can point out special properties of certain solids that lend themselves to interesting uses in sculpture: the nose of the hippo in Figure 17, for instance, is produced by removing a set of faces from a small rhombicosidodecahedron—an operation that, for this particular solid (and these particular faces), produces a new shape with a decagonal face. (See Figure 23.) In a similar vein, shapes such as the cuboctahedron and icosidodecahedron may be "sliced in two" to produce half-shapes (the first including an equatorial hexagon that can be seen in the original shape, the second an equatorial decagon [Coxeter, 1973]); such operations are staples of our own polyhedral sculptural design. More generally, polyhedral sculpture provides a fertile context in which to explore patterns in polyhedra: how they can be stretched, sliced, or squashed; whether they will stand on a surface; how they look in combination; how difficult or easy they are to construct; what decorative strategies they lend themselves to; and so forth. Unlike many "standard" exercises in solid geometry, the activities that we are currently constructing will encourage students to reflect upon polyhedra both from a mathematical perspective and from the "engineering" perspective inherent in sculptural work.

9 A tool similar in operation to the one proposed here was developed for an earlier Scheme-based application named SchemeChart [Eisenberg and Fischer, 1994].

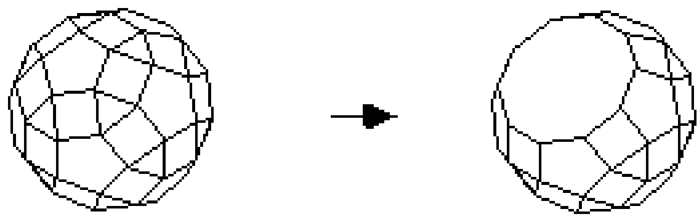

Figure 23. A small rhombicosidodecahedron (left) has a group of 11 faces removed to produce a new shape with a decagonal face (right).

5.2 Related work

In the previous section, we noted that there is a long and beautiful tradition of books on the subject of building solids from paper. Typically, these books focus on polyhedral modelling for "classical" shapes (rather than the sorts of customized shapes typical of HyperGami constructions), and those shapes are discussed strictly with regard to their mathematical (as opposed to artistic or engineering) properties. In this same educational tradition, one may place a variety of polyhedron-building kits (such as the marvelous Zometool [BioCrystal, Inc.]). Creating shapes using such kits is generally a quicker and less painstaking process than that of making HyperGami solids—the kits employ pieces that can be composed relatively easily. Still, these kits are generally limited (through the particular pieces that they employ) in the types of polyhedra that one can construct; and they lack the decorative possibilities exploited in HyperGami's orihedra.

HyperGami is not the only software application that includes some notion of presenting solid shapes and accompanying folding nets. One three-dimensional graphics application, *form·Z* [S1], includes an "unfolding" tool that permits the user to produce folding nets of solid objects; an educational application, *Tabs+* [S4], allows the user to create a variety of solids and their accompanying nets; and one elementary-school-level application named *Shape Up!* [S3] likewise includes folding nets for a selection of predefined solids. Nor are we the only practitioners of polyhedral sculpture: Suzuki [1987], as far as we know, was the first practitioner of using polyhedral elements to create larger structures (though his elements were largely undecorated and derived from a small set of "building block" shapes); and the aforementioned *Tabs+* application has been used to make polyhedral sculptures as well. To our knowledge, HyperGami is unique in its combination of (a) a focus both on classical polyhedra and rich collections of variants thereof; (b) the use of those elements for a variety of geometric papercrafting activities (including the creation of orihedra); and (c) its inclusion of an interactive programming environment for advanced work both in creating and decorating solids.

This last point was touched upon in the second section in this paper, in our discussion of the notion of "programmable applications" [Eisenberg, 1991]. Applications of this type seek to combine the respective strengths of direct manipulation interfaces (learnability, accessibility, aesthetic appeal) with those of programming (extensibility and expressive range). In this respect, HyperGami shares with such software efforts as LEGO / Logo [Resnick, 1993] an interest in embedding programming languages within design applications; it likewise shares with that effort an interest in integrating computational and "real-world" elements. In the case of LEGO / Logo, programming constructs are used to produce behaviors of real-world robotic "creatures"; in HyperGami, programming constructs are used to design the physical structure and appearance of tangible three-dimensional objects.

5.3 Ongoing and Future Work

Our work in orihedra construction over the past two years has served, for us, as a foundation from which to explore a wide range of research topics. In recent months we have embarked on several new projects, all based in part on this earlier effort:

• <u>Combining children's literature with mathematical design.</u> One project that we are currently pursuing involves the construction of "mathematical children's literature" combining both fantasy elements and interactive computational media for mathematical design (such as HyperGami). Our first effort in this direction was an alphabet book of paper sculpture, "AlphaBetaHedra": a draft of this book was completed earlier this year. (Several of the sculptures shown earlier—the mushrooms, hippopotamus, fish, and caterpillar—were all designed for this book.) Currently, we are developing a second children's book with a more pronounced narrative structure: this book will include scenes built out of polyhedral forms which can then be constructed anew (or altered) by the book's readers. Our intent in this effort is to provide an engaging introduction to mathematical design—an introduction that is (in the spirit of HyperGami orihedra) less abstract and decontextualized than children's mathematical activities often are. At the same time, we see this type of "computational children's literature" as constituting something of a response to the popular animated children's stories implemented on CD-ROMs; while these animated stories, in our view, de-emphasize reflective activity by rendering the reading experience closer to that of watching television, we hope to develop computational literature that is, if anything, more reflective and contemplative in spirit than printed literature.

• Puzzles and Toys as Constructible Objects. Another direction that we are pursuing involves a focus on mathematical and scientific toys. Here, the notion is that these objects—often marketed in the form of pre-constructed puzzles or science kits—may be designed by users themselves, with an eye both toward scientific content and artistic value. As an initial example of this idea, we have employed HyperGami to create a decorated seven-piece puzzle cube; more recently we have made an orihedral physics toy, shown in Figure 24. This is a paper sculpture of a pig that (because it is constructed with weights in its legs) is capable of balancing athwart a thread. Our goal in this effort, then, is to create crafting activities (including the construction of HyperGami figures) through which both children and adults may engage in the creation of scientific objects and mathematical manipulatives.

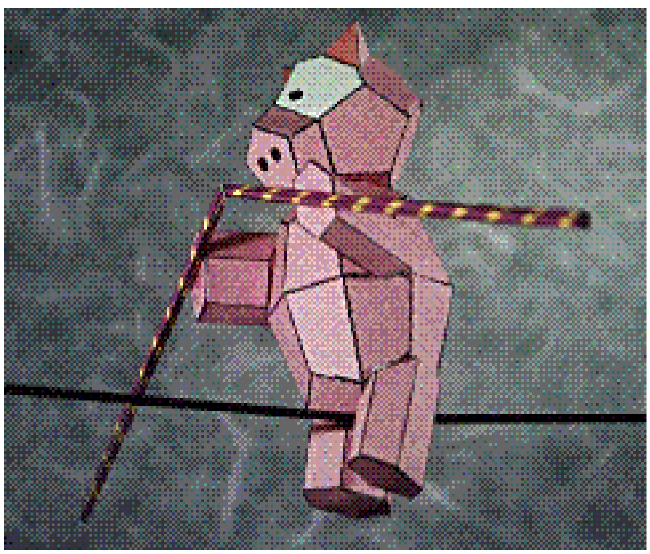

Figure 24. A HyperGami physics toy: a pig sculpture balanced on a string.

• Mechanical Design in Paper. One arguable limitation of orihedra is that they are static objects. Consequently, we have begun to explore the use of a design medium such as HyperGami to create dynamic, mechanical paper sculptures. Constructible working paper instruments can occasionally be seen in marketed form (e.g., [Rudolph 1983], [Spooner 1986]); but, again, these instruments are portrayed as recipes to be followed, rather than as instances of a creative medium in which the user can participate as an equal. In this light, we would like to expand the use of HyperGami (or, perhaps, new systems constructed along similar lines) so that they can be used to facilitate the creation of brand-new homemade paper machines. As an example of what such devices could look like, we have created several working paper machines composed of HyperGami polyhedra. One such machine, "Scheherezade", is shown in Figure 25: by turning the handle seen toward the right of the photograph, one can cause the small polyhedral figure on the magic carpet (at upper left) to move back and forth in a graceful rocking motion.[10] The essence of this machine's operation is the manner in which the circular motion of the peg at bottom left of the photograph is turned into the (more or less) linear motion of the rod supporting the little figure.

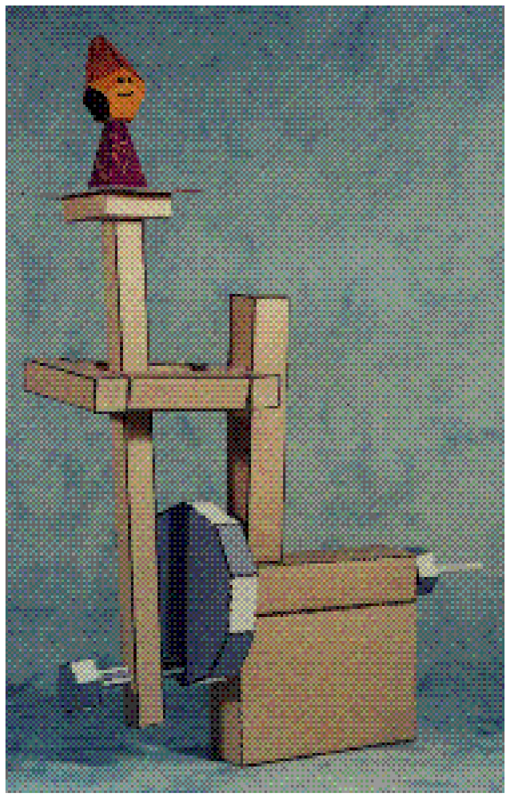

Figure 25. Scheherezade: a working paper machine.

• Working in materials other than paper. In part as a result of our increasing interest in constructing puzzles, games, toys, and machines, we have concurrently come to recognize the limitations of paper as a construction medium. HyperGami constructions have been made using different types of paper (such as cardstock) to achieve distinct structural or aesthetic effects; we have also begun to explore the use of HyperGami to work with materials other than paper. Figure 26 depicts several examples of work along these lines: the photograph shows polyhedral figures created in wax, plaster, and chocolate, all created by pouring the material

[10] It should perhaps be mentioned that the axle of this mechanical device is the only portion of the construction that is not a HyperGami solid (it is made from a paper lollipop stick).

in question into a HyperGami-created polyhedral mold. One promising direction for research, then, would be to design computational tools (perhaps based on HyperGami) that would act as "design assistants" for working in a variety of materials; for example, a user constructing a machine along the lines of the one in Figure 25 might want the program to offer advice on which portions of the device to construct (e.g.) in paper, cardstock, balsa wood, or plastic.

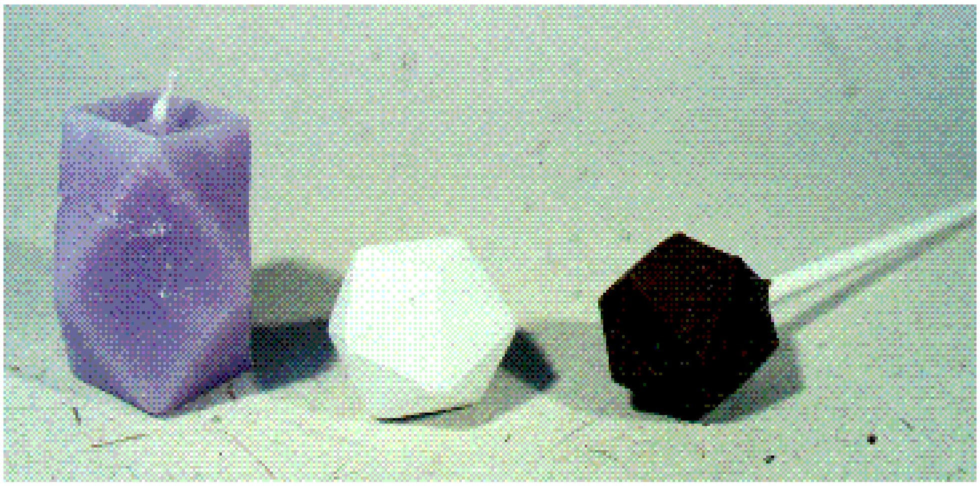

Figure 26. HyperGami polyhedra in other materials: wax (left), plaster (center), chocolate (right).

Acknowledgments
Three (anonymous) reviewers had extremely helpful and thought-provoking criticism of an earlier draft of this paper; we wish to thank them. Our special gratitude to Hal Abelson, Barry Dworkin, Eugene Hoffman, Mike Mills, and Jim Spohrer for support, inspiration, and great conversations. Zach Nies and Jay Smith did marvelous work extending HyperGami. Julie DiBiase has developed curricular materials for HyperGami, and Joan Hoffman made creative use of the system in constructing a collection of toroids. Thanks to Gerhard Fischer and the Center for LifeLong Learning and Design at the University of Colorado, who have provided a home for this research. This work has been supported in part by the National Science Foundation under grants RED-9253425 and REC-961396, and a Young Investigator award IRI-9258684. The second author is supported by a fellowship from the National Physical Science Consortium. Work described in the final section is supported in part by the National Science Foundation and the Advanced Research Projects Agency under Cooperative Agreement No. CDA-9408607. Finally, we would like to thank Apple Computer, Inc. for donating the machines with which this research was conducted.

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Software

[S1] *Form·Z*. Auto·Des·Sys, Inc. Columbus, Ohio.
[S2] *MacScheme*. Lightship Software. Available through Academic Distributing, Dewey AZ.
[S3] *Shape Up!* Sunburst Communications, Inc. Pleasantville, NY.
[S4] *Tabs+*. *The Knowledge Tree*. Toronto, Ontario, Canada.