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Lower Dimensional Interpolation in Overlapping Composite Mesh Difference Methods

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Introduction

We propose a modified Composite Mesh Difference Method (CMDM) in which a lower dimensional interpolation can be used along the interface of the nonmatching grids. The advantage of this approach is that fewer interpolation points are needed while the same order of global accuracy is preserved. This is important especially for distributed memory implementations since smaller amounts of data need to be communicated among the overlapping subdomains. A CMDM on two subdomains has been described by Starius [Sta77], while Cai et al. [CMS98] have studied the case of many subdomains. We focus on the 2D Poisson's equation. Our results show that it is possible to obtain global second order accuracy on nonmatching grids with a local coupling using only 4 points at the interface in the discretisation equations. In contrast, the overlapping nonmatching mortar method [CDS98] requires a global mortar projection involving all the mesh points on the interface.

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Composite Mesh Difference Method

We briefly describe our CMDM for solving the second order elliptic partial differential equation $\mathcal{L}u = f$ in Ω with a Dirichlet boundary condition $u = g$ on $\partial\Omega$. Given a domain Ω consisting of p nonoverlapping subdomains Ω_i such that $\bar{\Omega} = \cup_{i=1}^{p} \bar{\Omega}_i$, we independently construct a grid of size h_i on each of a set of enlarged subdomains Ω'_i. Due to the enlargement of the subdomains these grids overlap. We denote by $\Gamma_i = \partial\Omega'_i \cap \partial\Omega$ the intersection of the boundaries $\partial\Omega'_i$ and $\partial\Omega$.
Assumption 1 *The truncation error* $\alpha_i(x) = ((\mathcal{L}_{h_i} u_{h_i}) - (\mathcal{L}u))(x)$ *is of order* p_i*:*

$$\|\alpha_i\|_\infty \leq C_{\alpha_i} h_i^{p_i} \|u\|_{p_i+2,\infty,\Omega'_i}, \tag{1.1}$$

where C_{α_i} *is a constant independent of the mesh size* h_i *and* $\|u\|_{k,\infty,\Omega'_i}$ *denotes the Sobolev norm for the space* $W^k_\infty(\Omega'_i)$.
Assumption 2 *The interpolation operator* $\mathcal{I}_i$ *only uses values from* $\cup_{j\neq i}\bar{\Omega}_j$ *and no values from* Ω'_i*. The interpolation error* $\beta_i(x) = (u - \mathcal{I}_i u)(x)$ *is of order* q_i*:*

$$\|\beta_i\|_\infty \leq C_{\beta_i} h_i^{q_i} \|u\|_{q_i,\infty}. \tag{1.2}$$

The *interpolation constant* σ_i is the infinity norm $\sigma_i = \|\mathcal{I}_i\|_\infty$ of the matrix $\mathcal{I}_i$ representing the interpolation. If piecewise linear or bilinear interpolation is used, the interpolation constant $\sigma_i = 1$ is optimal. For quadratic or cubic interpolation we have $\sigma_i = 5/4$ in the 1D case and $\sigma_i = 25/16$ in the 2D case. We denote by $\sigma = \max_i \sigma_i$ the largest of all the interpolation constants.

The *global discretisation* $u_h = (u_{h_1}, u_{h_2}, \cdots, u_{h_p})$ on the composite grid is obtained by coupling the local discretisations through the requirement that the solution matches the interpolation of the discrete solution from adjacent grids. The system of equations consists of p subproblems, each having the following form:

$$\begin{cases} \mathcal{L}_{h_i} u_{h_i} = f_{h_i} & \text{in } \Omega'_i, \\ u_{h_i} = g_{h_i} & \text{on } \Gamma_i, \\ u_{h_i} = z_{h_i} = \mathcal{I}_i u_h & \text{on } \partial\Omega'_i \setminus \Gamma_i. \end{cases} \tag{1.3}$$

Assumption 3 *The local finite difference discretisations (1.3) are stable in the maximum norm and satisfy a strong discrete maximum principle, i.e. a constant* K_i *independent of* h_i *and a constant* $0 \leq \rho_i < 1$ *exist so that*

$$\|u_{h_i}\|_{\infty,\bar{\Omega}_i} \leq K_i \|f_{h_i}\|_{\infty,\Omega'_i} + \max\{\|g_{h_i}\|_{\infty,\Gamma_i}, \rho_i \|z_{h_i}\|_{\infty,\partial\Omega'_i\setminus\Gamma_i}\}. \tag{1.4}$$

The *contraction factor* $0 \leq \rho_i < 1$ measures the error reduction. This can be seen by considering the homogeneous problem, i.e. $f = 0$ and $g = 0$.
Assumption 4 *The product of the interpolation constant and the contraction factor is less than 1*

$$\tau = \max_i (\rho_i \sigma) < 1. \tag{1.5}$$

Under the above assumptions Cai et al. [CMS98] proved the maximum norm stability of the global discretisation and showed that the following error bound holds.

Theorem 1 *The error in the discrete solution satisfies*

$$\sum_{i=1}^{p} \|e_{h_i}\|_\infty \leq \left(1 + \frac{\sigma}{1-\tau}\right) \left(\sum_{i=1}^{p} K_i \|\alpha_i\|_\infty + \sum_{i=1}^{p} \|\beta_i\|_\infty\right). \tag{1.6}$$

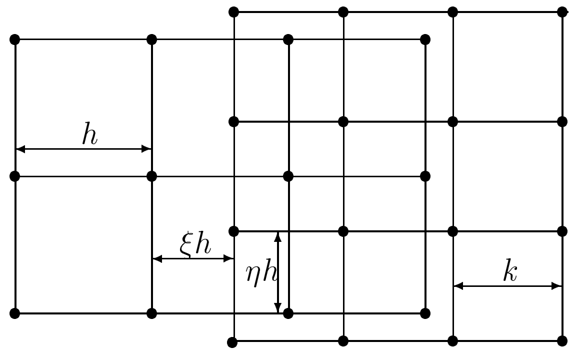

Figure 1.1 The scaled local coordinates (ξ, η) used in the interpolation.

Schwarz Alternating Method

Since the CMDM described above is a contraction mapping, the resulting system of equations can be solved by repeatedly solving the p subproblems (1.3) in parallel, where z_{h_i} is computed from the previous iteration. The convergence rate of this iteration is bounded by the contraction factor τ of the mapping.

Theorem 2 *The iterates* $\{u_h^{(n)}\}$ *converge to the exact discrete solution* u_h *and*

$$d(u_h^{(n)}, u_h) \leq \tau^n d(u_h^{(0)}, u_h). \tag{1.7}$$

This is a parallel variant of the Schwarz alternating method. Rather than using the additive Schwarz method as a solver, it is better to use it as a preconditioner in a Krylov subspace method. Convergence proofs of the Schwarz algorithm based on a maximum principle can be found in [CMS98, Mil65].

We remark that when solving (1.3) using iterative methods, the total arithmetic cost is determined by the subdomain mesh sizes, while the communication cost, for implementations on distributed memory machines, depends on the interface interpolation operators. The rest of this paper is devoted to the interpolation issue.

Interface Interpolation Schemes

In this section, we discuss several interface interpolation schemes. We restrict ourselves to the two subdomains case, i.e. $\Omega = \Omega_1' \cup \Omega_2'$ where $\Omega_1' = [0, l_1] \times [0, 1]$ and $\Omega_2' = [l_2, 2] \times [0, 1]$. We assume $l_1 > 1$ and $l_2 < 1$. The usual five-point finite difference method is used in the two subdomains.

The standard stencil with bilinear interpolation

Since both the standard five-point stencil and bilinear interpolation are second order, the error bound (1.6) shows that the resulting CMDM is also second order. However this scheme does not satisfy the *consistent interpolation* condition defined by Goossens et al. [GCR98]. We describe the inconsistency present in this approach. Let $(0,0)$ be the local coordinates of a mesh point in Ω_1' that is next to the interface. To define the finite difference stencil, the value at $(h,0)$ has to be obtained from Ω_2 through interpolation. More precisely, the stencil S at $(0,0)$ has the form

$$S = -4u(0,0) + u(0,-h) + u(0,h) + u(-h,0) + v, \tag{1.8}$$

where v is computed using a bilinear interpolation for $u(h,0)$, i.e. $v = (1-\xi)(1-\eta)u(h-\xi k, -\eta k) + (1-\xi)\eta u(h-\xi k, (1-\eta)k) + \xi(1-\eta)u(h+(1-\xi)k, -\eta k) + \xi\eta u(h+(1-\xi)k, (1-\eta)k)$. Figure 1.1 shows the scaled local coordinates (ξ,η) used in the interpolation on the overlapping meshes. Expanding (1.8) we find the inconsistent discretisation for the nodes where the bilinear interpolation is used:

$$\frac{S}{h^2} - (u_{xx} + u_{yy}) = \frac{\gamma_k^2}{2}\left(\xi(1-\xi)u_{xx} + \eta(1-\eta)u_{yy}\right) + \mathcal{O}(h), \tag{1.9}$$

where $\gamma_k = k/h$ is the ratio of the mesh sizes. Note that the scheme is consistent only if ξ and η are either 0 or 1, which implies that the two meshes match each other on the interface. We show numerical results illustrating the effect of this inconsistent discretisation.

Theorem 3 *The standard stencil with bilinear interpolation results in a second order scheme on every enlarged subdomain Ω_i':*

$$|e_p| \le C_{\Phi_a}\, max\left\{C_I, max_{p_j\in\mathcal{J}}\left(\xi(1-\xi)u_{xx} + \eta(1-\eta)u_{yy}\right)\frac{\gamma_k^2}{2\tilde{E}_2}\right\} h^2 + C_D h^2 + \mathcal{O}(h^3), \tag{1.10}$$

$\forall p \in \Omega_i'$, where $C_I = (M_{xxxx} + M_{yyyy})/(48E_1)$ is a constant depending on the derivatives u_{xxxx} and u_{yyyy}. The constant C_{Φ_a} denotes the maximum of the nonnegative function Φ over the grid points where a Dirichlet boundary condition is used for the subdomain Ω_i' and $C_D \ge 0$ is a constant associated with the accuracy of the Dirichlet boundary conditions on the internal boundaries $\partial\Omega_i' \setminus \Gamma_i$. The set $\mathcal{J}$ contains all the points where the inconsistent discretisation (1.9) is used. The constants E_1 and $\tilde{E}_2$ are lower bounds for $\mathcal{L}_{h_i}\Phi$.

Second Order Scheme on a Modified Stencil

In order to obtain a consistent discretisation, we construct a second order accurate difference formula on the modified stencil depicted in Fig. 1.2. The idea is to slightly modify the meshes along their boundaries so that the interpolation is needed only in one of the directions in contrast to the standard bilinear method, which needs information in both x and y directions. The mesh width $k \ge h$ is selected so that a grid line in the other mesh is matched and no interpolation along the x-axis is required. The point $u(-2h,0)$ is needed to obtain a second order discretisation in the stencil

$$S = c_{0,0}u(0,0) + c_{0,-1}u(0,-h) + c_{0,1}u(0,h) + c_{-1,0}u(-h,0) + c_{1,0}u(k,0) + c_{-2,0}u(-2h,0) \tag{1.11}$$

with $c_{0,0} = -1 - 3/\gamma_k$, $c_{0,-1} = c_{0,1} = 1$, $c_{-1,0} = 2(2-\gamma_k)/(\gamma_k+1)$, $c_{1,0} = 6/(\gamma_k(\gamma_k+1)(\gamma_k+2))$ and $c_{-2,0} = (\gamma_k-1)/(\gamma_k+2)$. The truncation error is given as

$$\frac{S}{h^2} - (u_{xx} + u_{yy}) = \frac{h^2}{12}\left((3\gamma_k - 2)u_{xxxx} + u_{yyyy}\right) + \mathcal{O}(h^3). \tag{1.12}$$

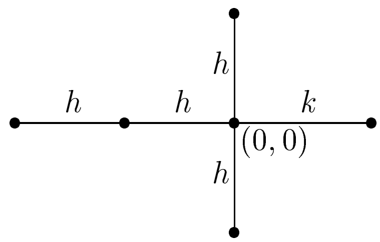

Figure 1.2 Modified stencil for the second order scheme.

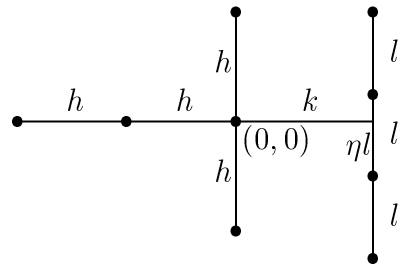

Figure 1.3 Stencil for the second order scheme with 1D cubic interpolation.

Note that for $1 \leq \gamma_k \leq 2$ this stencil satisfies a discrete maximum principle.

Modified Stencil with Linear Interpolation

We now consider the effect of replacing $u(k,0)$ by an interpolation formula along the y-axis. First of all we show that a consistent approximation exists when linear interpolation is used, i.e. $v = (1-\eta)u(k,-\eta l) + \eta u(k,(1-\eta)l)$ is used instead of $u(k,0)$. We use the standard stencil and seek the coefficients in

$$S = c_{0,0}u(0,0) + c_{0,-1}u(0,-h) + c_{0,1}u(0,h) + c_{-1,0}u(-h,0) + c_v v \qquad (1.13)$$

so that a consistent approximation to $(u_{xx} + u_{yy})$ results. Setting $c_{0,-1} = c_{0,1} = 1 - \eta(1-\eta)\gamma_l^2/(\gamma_k(\gamma_k+1))$, $c_{-1,0} = 2/(\gamma_k+1)$, $c_v = 2/(\gamma_k(\gamma_k+1))$ and $c_{0,0} = 2\eta(1-\eta)\gamma_l^2 - (\gamma_k+1)^2/(\gamma_k(\gamma_k+1))$, where $\gamma_l = l/h$, results in

$$\frac{S}{h^2} - (u_{xx} + u_{yy}) = \mathcal{O}(h). \qquad (1.14)$$

If $\gamma_l \leq 2\sqrt{\gamma_k(\gamma_k+1)}$ this formula satisfies a discrete maximum principle. For η equal to 0 or 1 a first order discretisation on a modified five-point stencil is obtained. Note that the difference formula has to be modified to account for the low order interpolation.

Modified Stencil with Cubic Interpolation

The interpolation along the x-axis can be avoided by using a modified stencil. We show that cubic interpolation along the y-axis results in a second order scheme. This is equivalent to constructing a second order accurate difference formula on the modified stencil depicted in Fig. 1.3.

Theorem 4 *It is not possible to obtain a second order accurate discretisation of $(u_{xx} + u_{yy})$ at the center point $(0,0)$ if only 3 or fewer points are used along the line $x = k$ and none of these points has a zero y-coordinate.*

Theorem 5 *The only second order accurate discretisation of $(u_{xx}+u_{yy})$ at the center point $(0,0)$ using only 4 points along the line $x = k$ with none of these points having a*

Table 1.1 Results for the standard stencil with bilinear interpolation.

l	ξ	$\|e_{\Omega'_1}\|_\infty$	ratio	$\|e_{\Omega'_2}\|_\infty$	ratio
0	0.6	5.53e-2		6.34e-2	
1	0.2	1.40e-2	3.94	1.31e-2	4.82
2	0.4	3.62e-2	3.88	3.47e-3	3.80
3	0.8	8.79e-4	4.11	8.26e-4	4.19
4	0.6	2.26e-4	3.89	2.16e-4	3.82
5	0.2	5.50e-5	4.11	5.17e-5	4.18
6	0.4	1.41e-5	3.89	1.35e-5	3.83

zero y-coordinate, is the second order scheme (1.11) on the modified stencil with cubic interpolation along the line $x = k$ for the point $(k, 0)$.

We seek the coefficients in the stencil

$$\begin{aligned} S \;=\; & c_{0,0}u(0,0) + c_{0,-1}u(0,-h) + c_{0,1}u(0,h) + c_{-1,0}u(-h,0) + c_{-2,0}u(-2h,0) + \\ & c_{1,-1}u(k,-(1+\eta)l) + c_{1,0}u(k,-\eta l) + c_{1,1}u(k,(1-\eta)l) + c_{1,2}u(k,(2-\eta)l). \end{aligned} \tag{1.15}$$

Setting $c_{0,0} = -1 - 3/\gamma_k$, $c_{0,-1} = c_{0,1} = 1$, $c_{-1,0} = 2(2-\gamma_k)/(\gamma_k+1)$, $c_{-2,0} = (\gamma_k - 1)/(\gamma_k + 2)$, $c_{1,-1} = -\eta(1-\eta)(2-\eta)/N$, $c_{1,0} = 3(1-\eta)(2-\eta)(\eta+1)/N$, $c_{1,1} = 3(2-\eta)\eta(\eta+1)/N$ and $c_{1,2} = -\eta(1-\eta)(\eta+1)/N$, where $N = \gamma_k(\gamma_k+1)(\gamma_k+2)$ results in a second order accurate discretisation

$$\frac{S}{h^2} - (u_{xx} + u_{yy}) = \mathcal{O}(h^2). \tag{1.16}$$

The truncation error for this scheme is

$$\alpha = \left[\frac{(3\gamma_k - 2)}{12}u_{xxxx} + \left(\frac{1}{12} - \frac{(2-\eta)(1-\eta)\eta(\eta+1)\gamma_l^4}{4\gamma_k(\gamma_k+1)(\gamma_k+2)}\right)u_{yyyy}\right]h^2 + \mathcal{O}(h^3). \tag{1.17}$$

It is clear that the coefficients $c_{1,-1}$, $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ are equal to the product of the coefficient $c_{1,0}$ in (1.11) and the cubic Lagrange interpolation polynomials.

Numerical results

Our testcase is taken from [CMS98] and concerns the solution of $-\nabla^2 u = f$ on $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [0,1] \times [0,1]$ and $\Omega_2 = [1,2] \times [0,1]$. The r.h.s. f and the Dirichlet boundary conditions g are chosen so that the exact solution is $u(x,y) = (\sin(\pi x) + \sin(\pi x/2))\sin(\pi y)$. The overlapping subdomains are $\Omega'_1 = [0,1.4] \times [0,1]$ with $h_1 = 0.2 \times 2^{-l}$ and $\Omega'_2 = [0.75,2] \times [0,1]$ with $h_2 = 0.25 \times 2^{-l}$.

In Table 1.1 we list the L_∞-norm of the error for the standard stencil with bilinear interpolation. For a second order scheme, the ratio between two successive error norms should be 4 when the mesh size is halved. These results show ratios alternating between 4.11 and 3.89 and between 4.18 and 3.83! This is due to the presence of the inconsistency as shown by (1.9) which results in a dependency of the error on $\xi(1-\xi)$, i.e. the relative position of the interface in the other mesh. For this testcase the dominant term in the error bound (1.10) is $e \approx (\xi(1-\xi)c_1 + c_2)\,h^2$, where c_1 and

Table 1.2 Results for standard stencil with bicubic interpolation (columns 2–5) and for modified stencil with 1D cubic interpolation (columns 6–9).

l	$\|e_{\Omega_1'}\|_\infty$	ratio	$\|e_{\Omega_2'}\|_\infty$	ratio	$\|e_{\Omega_1'}\|_\infty$	ratio	$\|e_{\Omega_2'}\|_\infty$	ratio
0	5.20e-2		4.14e-2		4.98e-2		8.41e-2	
1	1.34e-2	3.885	1.19e-2	3.4810	1.34e-2	3.7168	1.51e-2	5.5828
2	3.33e-3	4.0164	3.01e-3	3.9605	3.31e-3	4.0523	3.67e-3	4.0997
3	8.34e-4	3.9986	7.50e-4	4.0072	8.34e-4	3.9687	8.22e-4	4.4717
4	2.08e-4	4.0010	1.88e-4	4.0012	2.08e-4	4.0017	1.95e-4	4.2099
5	5.21e-5	4.0003	4.69e-5	4.0004	5.21e-5	3.9999	4.78e-5	4.0868
6	1.30e-5	4.0000	1.17e-5	3.9998	1.30e-5	4.0004	1.18e-5	4.0393

c_2 are constants independent of ξ and h. With this expression, we can estimate the ratio γ_e between two successive error norms. When the mesh is refined by halving the mesh size, i.e. $h_{i+1} = h_i/2$, we have

$$\gamma_e = \frac{\|e_{\Omega'_{h_i}}\|_\infty}{\|e_{\Omega'_{h_{i+1}}}\|_\infty} = \frac{c_1\,(\xi_i(1-\xi_i)+\gamma_c)\,h_i^2}{c_1\,(\xi_{i+1}(1-\xi_{i+1})+\gamma_c)\,h_{i+1}^2} = \frac{\xi_i(1-\xi_i)+\gamma_c}{\xi_{i+1}(1-\xi_{i+1})+\gamma_c}\,4 \qquad (1.18)$$

where $\gamma_c = c_2/c_1$. For Ω_1' we find $\gamma_c = 2.7491$ which results in ratios γ_e of 4.11 and 3.89 while for Ω_2' we have $\gamma_c = 1.6178$ resulting in ratios γ_e of 4.18 and 3.83. The accuracy of the scheme depends on the relative position of the interface in the other mesh. Apart from this phenomenon the scheme is second order, since fitting a power of the mesh size $\|e_{\Omega_1'}\|_\infty \approx \kappa h^\lambda$ yields $\lambda = 1.9929$. The second order accuracy can also be seen when the mesh is refined twice, i.e. the mesh size is divided by 4, in this case the factor $\xi(1-\xi)$ does not change and we get ratios of 15.9973 (for $\|e_{\Omega_1'}\|_\infty$) and of 15.9966 (for $\|e_{\Omega_2'}\|_\infty$) between two successive error norms, which is very close to the theoretical value of 16.

In Table 1.2 we show the results for the standard stencil with bicubic interpolation, which is a very expensive interpolation since it requires 16 points, and for the modified stencil with 1D cubic interpolation, which only requires 4 points. It is clear that both these schemes are second order and that our method is as accurate as the classical approach with expensive bicubic interpolation.

In order to see the effect of the overlap, we fix the mesh sizes to be $h_1^{-1} = 320$ and $h_2^{-1} = 256$ and vary the overlap according to $\delta_1 = 2 \times 2^\chi h_1$ and $\delta_2 = 2^\chi h_2$ for the values of χ listed in Table 1.3. We list the number of additive Schwarz iterations required to satisfy the convergence criterion of $\|r_n\|_2 \leq 10^{-10}\|r_0\|_2$ and the L_∞-norm of the error in the nonoverlapping subdomains Ω_1 and Ω_2. From the results it is clear that the global accuracy of the first method increases as the overlap increases, thus necessitating substantial overlap, while our method reaches the attainable accuracy even with minimal overlap. As expected the number of additive Schwarz iterations decreases, as the overlap increases.

Table 1.3 Effect of overlap on the convergence rate of the Schwarz method and on the accuracy for the standard stencil with bilinear interpolation (columns 2–4) and for the modified stencil with 1D cubic interpolation (columns 5–7).

χ	n	$\|e_{\Omega_1}\|_\infty$	$\|e_{\Omega_2}\|_\infty$	n	$\|e_{\Omega_1}\|_\infty$	$\|e_{\Omega_2}\|_\infty$
0	846	4.00e-4	4.00e-4	369	1.31e-5	6.49e-6
1	436	1.89e-4	1.90e-4	300	1.31e-5	6.49e-6
2	224	7.99e-5	7.93e-5	161	1.31e-5	6.48e-6
3	116	3.12e-5	3.19e-5	96	1.31e-5	6.48e-6
4	60	9.97e-6	1.04e-5	53	1.30e-5	6.51e-6
5	32	1.24e-5	7.47e-6	30	1.30e-5	6.73e-6
6	17	1.41e-5	6.82e-6	16	1.30e-5	7.52e-6
7	10	1.33e-5	6.53e-6	9	1.31e-5	9.19e-6

Concluding remarks

We have studied several interface interpolation schemes for overlapping nonmatching grids finite difference methods. A scheme based on the combination of 1D cubic interpolation and a six-point stencil is proposed and produces a consistent and globally second order method. We have also shown numerically that minimum overlap is required to achieve the accuracy, and that larger overlap reduces the number of Schwarz iterations but does not change the accuracy. The method is cheaper than the interpolation method required by the theory of [CH90] and the mortar based method proposed in [CDS98].

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