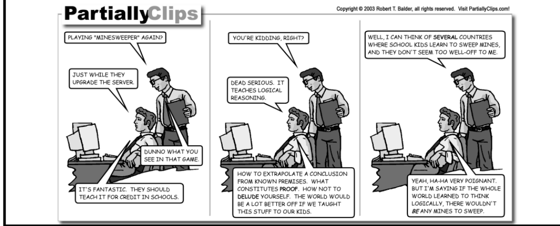
Automated Theorem Proving and Proof Checking

Meeting 26, CSCI 5535, Spring 2009



Announcements

- Class Wednesday
  - Last class before presentations
  - FCQ during class
    - Please attend and give your feedback
    - Want 100% returns
  - What do you want with your last class?
    - Shape analysis (my job talk!)
    - Object-oriented calculus
    - Concurrency calculus
    - ?
- Presentations: send me title by Thursday
  - Take a look at evaluation form on website

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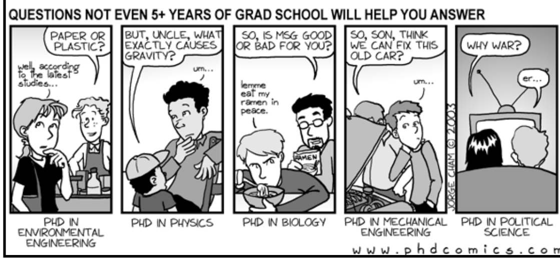
Review of Types for Data Abstraction



Data Abstraction

- It is useful to separate the creation of the abstract type and its use (newsflash …)
- Extend the syntax (t = imp, σ = interface):
  Terms ::= … | < t = τ, e : σ > | open e_a as t, x : σ in e_b
  Types ::= … | ∃t. σ
- The expression <t=τ, e : σ> takes the concrete implementation e and "packs it" as a value of an abstract type
  - Alternative notation: "pack e as ∃t. σ with t = τ"
  - "existential types" – used to model the stack, etc.
- The "open" expression allows e_b to access the abstract type expression e_a using the name x, the unknown type of the concrete implementation "t" and the interface σ

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Typing Rules for Existential Types

- Use Γ to also track type variables

$$\frac{\Gamma \vdash e : [\tau/t]\sigma}{\Gamma \vdash \langle t = \tau, e : \sigma \rangle : \exists t.\sigma}$$

$$\frac{\Gamma \vdash e_a : \exists t.\sigma \quad \Gamma, t, p : \sigma \vdash e_b : \tau}{\Gamma \vdash \text{open } e_a \text{ as } t, p : \sigma \text{ in } e_b : \tau} \quad t \notin \mathrm{FV}(\Gamma \cup \tau)$$

- The restriction in the rule for "open" ensures that t does not escape its scope

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Modularity

- A module is a program fragment along with **visibility constraints**
- Visibility of functions and data
  - Specify the function interface but hide its implementation
- Visibility of type definitions
  - More complicated because the type might appear in specifications of the visible functions and data
  - Can use data abstraction to handle this
- A module is represented as a type component and an implementation component
  <t = τ, e : σ> (where t can occur in e and σ)
  - even though the specification (σ) refers to the implementation type we can still hide the latter
  - But there are problems …

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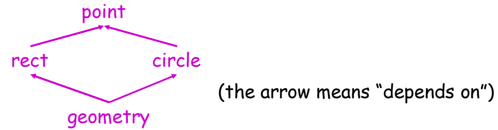
Existentials: The Good and Bad

- Existential types
  - Allow representation (type) hiding
  - Allow separate compilation. Need to know only the type of a module to compile its client
  - First-class modules. They can be selected at run-time. (cf. OO interface subtyping)
- Problems:
  - Closed scope. Must open an existential before using it!
  - Poor support for module hierarchies

Problems with Existentials

- There is an inherent tension between handling modules in isolation (good for separate compilation, interchangeability) and the need to integrate them



(the arrow means "depends on")

- Solution 1: open "point" at top level
  - Inversion of program structure
  - The most basic construct has the widest scope

Give Up Abstraction?

- Solution 2: incorporate point in rect and circle

  R = < point = …, <rect = point × point, …> … >

  C = < point = …, <circle = point × real, …> … >
- When we open R and C we get *two distinct notions of point*!
  - And we will *not* be able to combine them
- Another option is to allow the type checker to see the representation type
  - and thus give up representation hiding

Strong Sums

- New way to open a package

  Terms e ::= … | Ops(e)

  Types τ ::= … $\Sigma t.\tau$ | Typ(e)
  - Use Typ and Ops to decompose the module
  - Operationally, they are just like "fst" and "snd"
  - $\Sigma t.\tau$ is the dependent sum type
  - It is like $\exists t.\tau$ except we can look at the type

$$\frac{\Gamma \vdash e : \Sigma t.\tau}{\Gamma \vdash \mathsf{Ops}(e) : [\mathsf{Typ}(e)/t]\tau}$$

Modularity with Strong Sums

- Consider the R and C defined as before:

let Pt = <point = real × real, …> : Σpoint. τ_P in

let R = <point = Typ(Pt),

< rect = point × point, …> : Σrect. τ_R in

let C = <point = Typ(Pt),

< circle = point × real, …> : Σcircle. τ_C

- Since we use strong-sums the type checker sees that the two point types are the same

Modules with Strong Sums

- ML's module system is based on strong sums

Problems:

- Poorer data abstraction
- Expressions appear in types (Typ(e))
  - Types might not be known until at run time
  - Lost separate compilation
  - Trouble if e has side-effects (but we can use a value restriction – e.g., "IntSet.t")
- Second-class modules (because of value restriction)
- We can combine existentials with strong sums
  - Translucent sums: partially visible

End of Types for Data Abstraction

On to Automated Theorem Proving

Cunning Theorem-Proving Plan

- There are full-semester courses on automated deduction; we will elide details.
- Logic Syntax
- Theories
- Satisfiability Procedures
- Mixed Theories
- Theorem Proving
- Proof Checking
- SAT-based Theorem Provers

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Motivation

- Can be viewed as "decidable AI"
  - Would be nice to have a procedure to automatically reason from premises to conclusions …
- Used to rule out the exploration of infeasible paths (model checking, dataflow)
- Used to reason about the heap (McCarthy, symbolic execution)
- Used to automatically synthesize programs from specifications
- Used to discover proofs of conjectures (e.g., Tarski conjecture proved by machine in 1996, efficient geometry theorem provers)
- Generally under-utilized

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History

Automated deduction is *logical deduction performed by a machine*

- Involves logic and mathematics
- One of the oldest and technically deepest fields of computer science
  - Some results are as much as 75 years old
  - "Checking a Large Routine", Turing 1949
  - Automation efforts are about 40 years old
  - Floyd-Hoare axiomatic semantics
- Still experimental (even after 40 years)

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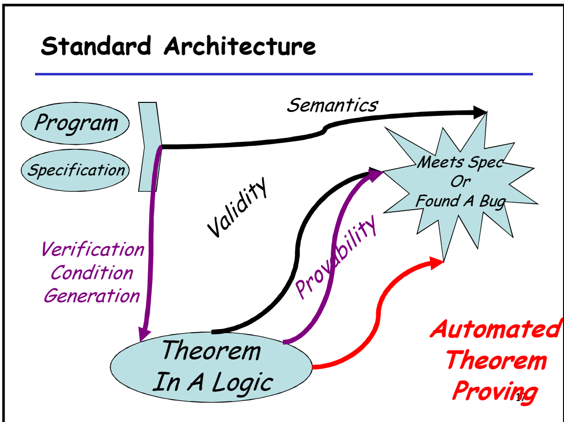
Standard Architecture



Logic Grammar

- We'll use the following logic:

Goals: $G ::= L \mid true \mid G_1 \wedge G_2 \mid H \Rightarrow G \mid \forall x.\ G$

Hypotheses: $H ::= L \mid true \mid H_1 \wedge H_2$

Literals: $L ::= p(E_1, \ldots, E_k)$

Expressions: $E ::= n \mid f(E_1, \ldots, E_m)$

- This is a subset of first-order logic
  - Intentionally restricted: no $\vee$ so far
  - Predicate functions p: <, =, …
  - Expression functions f: +, *, sel, upd,

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Theorem Proving Problem

- Write an algorithm "prove" such that:
- If prove(G) = true then $\models G$
  - Soundness (must have)
- If $\models G$ then prove(G) = true
  - Completeness (nice to have, optional)
- prove(H,G) means prove $H \Rightarrow G$
- Architecture: Separation of Concerns
  - #1. Handle $\wedge$, $\Rightarrow$, $\forall$, =
  - #2. Handle $\leq$, *, sel, upd, =

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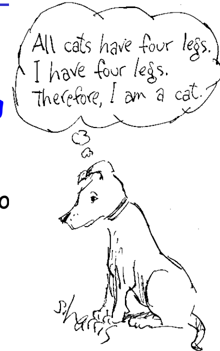
Theorem Proving

- Want to prove true things
- Avoid proving false things
- We'll do **proof-checking** later to rule out the "proof" shown here
- For now, let's just get to the point where we can prove something



Basic Symbolic Theorem Prover

- Let's define **prove(H,G)** …

prove(H, true) = **true**

prove(H, $G_1 \wedge G_2$) = prove (H, G_1) and prove (H, G_2)

prove(H_1, $H_2 \Rightarrow G$)= prove (H $\wedge H_2$, G)

prove(H, $\forall$x. G) = prove (H, [a/x]G) (a is fresh)

prove(H, L) = ???

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Basic Symbolic Theorem Prover

- Let's define **prove(H,G)** …

prove(H, true) = **true**

prove(H, $G_1 \wedge G_2$) = **prove(H,G_1) && prove(H, G_2)**

prove(H_1, $H_2 \Rightarrow G$)= **prove($H_1 \wedge H_2$, G)**

prove(H, $\forall$x. G) = **prove(H, G[a/x])** ***(a is "fresh")***

prove(H, L) = ???

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Theorem Prover for Literals

- We have reduced the problem to

  prove(H,L)

- But H is a conjunction of literals $L_1 \wedge \ldots \wedge L_k$
- Thus we really have to prove that

  $L_1 \wedge \ldots \wedge L_k \Rightarrow L$

- Equivalently, that $L_1 \wedge \ldots \wedge L_k \wedge \neg L$ is unsatisfiable
  - For any assignment of values to variables the truth value of the conjunction is false
- Now we can say

  prove(H,L) = Unsat(H $\wedge \neg$ L)

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Theory Terminology

- A theory consists of a set of functions and predicate symbols (*syntax*) and definitions for the meanings of those symbols (*semantics*)
- Examples:
  - **0, 1, -1, 2, -3, …, +, -, =, <** (usual meanings; "theory of integers with arithmetic" or "Presburger arithmetic")
  - **=**, $\leq$ (axioms of transitivity, anti-symmetry, and $\forall x.\ \forall y.\ x \leq y \vee y \leq x$; "theory of total orders")
  - **sel, upd** (McCarthy's "theory of arrays")

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Decision Procedures for Theories

- The Decision Problem
  - Decide whether a formula in a theory with first-order logic is true
- Example:
  - Decide "$\forall x.\ x>0 \Rightarrow (\exists y.\ x=y+1)$" in $\{\mathbb{N}, +, =, >\}$
- A theory is decidable when there is an algorithm that solves the decision problem
  - This algorithm is the decision procedure for that theory

Satisfiability Procedures

- The Satisfiability Problem
  - Decide whether a conjunction of literals in the theory is satisfiable
  - Factors out the first-order logic part
  - The decision problem can be reduced to the satisfiability problem
    - Parameters for $\forall$, skolem functions for $\exists$, negate and convert to DNF (sorry; I won't explain this here)
- "Easiest" Theory = Propositional Logic = SAT
  - A decision procedure for it is a "SAT solver"

Theory of Equality

- Theory of equality with uninterpreted functions (UIF)
- Symbols: $=, \neq$, **f**, **g**, …
- Axiomatically defined (A,B,C $\in$ Expressions):

$$\frac{}{A=A} \qquad \frac{B=A}{A=B} \qquad \frac{A=B \quad B=C}{A=C} \qquad \frac{A=B}{f(A) = f(B)}$$

- Example satisfiability problem:

$$g(g(g(x)))=x \wedge g(g(g(g(g(x)))))=x \wedge g(x)\neq x$$

Why is uninterpreted funcs important?

- Types
- Abstract anything as UIF

$x * y = y * x$??

$x * y = x * y$ (using UIF)

More Satisfying Examples

- Theory of Linear Arithmetic
  - Symbols: $\geq$, $=$, $+$, $-$, **integers**
  - Example: $y > 2x + 1, x > 1, y < 0$ is unsat
- Theory of Lists
  - Symbols: **cons**, **head**, **tail**, **nil**

$$\frac{}{head(cons(A,B)) = A} \qquad \frac{}{tail(cons(A,B) = B}$$

  - Theorem: $head(x) = head(y) \wedge tail(x) = tail(y) \Rightarrow x = y$

Mixed Theories

- Often we have facts involving *symbols from multiple theories*
  - E's symbols $=, \neq$, f, g, … (uninterp function equality)
  - R's symbols $=, \neq, +, -, \leq$, 0, 1, … (linear arithmetic)
  - Running Example (and Fact):
    $\models x \leq y \ \wedge\ y + z \leq x \ \wedge\ 0 \leq z \ \Rightarrow\ f(f(x) - f(y)) = f(z)$
  - To prove this, we must decide:
    Unsat($x \leq y,\ y + z \leq x,\ 0 \leq z,\ f(f(x) - f(y)) \neq f(z)$)
- We may have a sat procedure for each theory
  - E's sat procedure by Ackermann in 1924
  - R's proc by Fourier
- The sat proc for their combination is much harder
  - Only in 1979 did we get E+R

Satisfiability of Mixed Theories

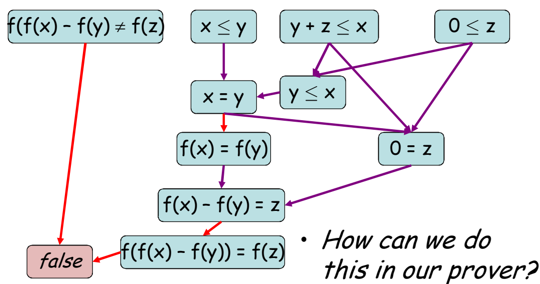
Unsat($x \leq y,\ y + z \leq x,\ 0 \leq z,\ f(f(x) - f(y)) \neq f(z)$)

- Can we just separate out the terms in Theory 1 from the terms in Theory 2 and see if they are separately satisfiable?
  - No, unsound, equi-sat ≠ equivalent.
- The problem is that the two satisfying assignments may be incompatible
- Idea (Nelson and Oppen): Each sat proc announces all equalities between variables that it discovers

Handling Multiple Theories

- We'll use cooperating decision procedures
- Each sat proc works on the literals it understands
- Sat procs share information (equalities)

Consider Equality and Arith

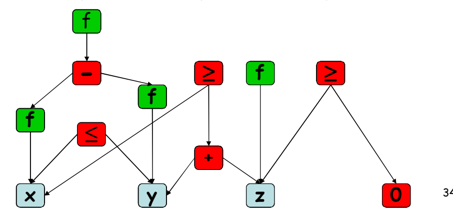


- *How can we do this in our prover?*

Nelson-Oppen: The E-DAG

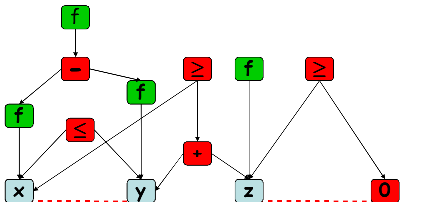
- Represent all terms in one Equivalence DAG
  - Node names act as variables shared between theories!

$f(f(x) - f(y)) \neq f(z) \wedge y \geq x \wedge x \geq y + z \wedge z \geq 0$



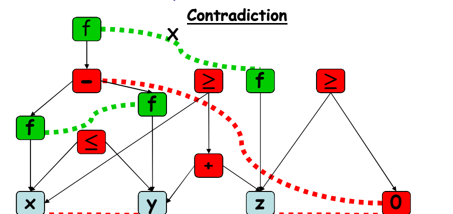
Nelson-Oppen: Processing

- Run each sat proc
  - Report all contradictions (as usual)
  - Report all equalities between nodes (key idea)



Nelson-Oppen: Processing

- Broadcast all discovered equalities
  - Rerun sat procedures
  - Until no more equalities or a contradiction



Does It Work?

- If a contradiction is found, then unsat
  - This is sound if sat procs are sound
  - Because only sound equalities are ever found
- If there are no more equalities, then sat
  - Is this complete? Have they shared enough info?
  - Are the two satisfying assignments compatible?
  - Yes!
  - *(Countable theories with infinite models admit isomorphic models, convex theories have necessary interpretations, etc.)*

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SAT-Based Theorem Provers

- Recall separation of concerns:
  - #1 Prover handles connectives ($\forall, \wedge, \Rightarrow$)
  - #2 Sat procs handle literals ($+, \leq, 0$, head)
- Idea: reduce proof obligation into propositional logic, feed to SAT solver (CVC)
  - To Prove: $3*x=9 \Rightarrow (x = 7 \wedge x \leq 4)$
  - Becomes Prove: $A \Rightarrow (B \wedge C)$
  - Becomes Unsat: $A \wedge \neg(B \wedge C)$
  - Becomes Unsat: $A \wedge (\neg B \vee \neg C)$

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SAT-Based Theorem Proving

- **To Prove: $3*x=9 \Rightarrow (x = 7 \wedge x \leq 4)$**
  - **Becomes Unsat: $A \wedge (\neg B \vee \neg C)$**
  - **SAT Solver Returns: A=1, C=0**
  - **Ask sat proc: unsat($3*x=9$, $\neg x \leq 4$) = true**
  - **Add constraint: $\neg(A \wedge \neg C)$**
  - **Becomes Unsat: $A \wedge (\neg B \vee \neg C) \wedge \neg(A \wedge \neg C)$**
  - **SAT Solver Returns: A=1, B=0, C=1**
  - **Ask sat proc: unsat($3*x=9$, $\neg x=7$, $x \leq 4$) = false**
    - **(x=3 is a satisfying assignment)**
  - **We're done! (original to-prove goal is false)**
  - **If SAT Solver returns "no satisfying assignment" then original to-prove goal is true**

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Proofs

Proof Generation

- We want our theorem prover to **emit proofs**
  - No need to trust the prover
  - Can find bugs in the prover
  - Can be used for proof-carrying code
  - Can be used to extract invariants
  - Can be used to extract models (e.g., in SLAM)
- Implements the soundness argument
  - On every run, a soundness proof is constructed

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Proof Representation

Proofs are trees
- Leaves are hypotheses/axioms
- Internal nodes are inference rules

Axiom: "true introduction"
- Constant: truei : pf
- pf is the type of proofs

Inference: "conjunction introduction"
- Constant: andi : pf → pf → pf

Inference: "conjunction elimination"
- Constant: andel : pf → Pf

Problem:
- "andel truei : pf" but does not represent a valid proof
- Need a more powerful *type system that checks content*

(handwritten: $\frac{}{true}$ truei; $\frac{A \quad B}{A \wedge B}$ andi; $\frac{A \wedge B}{A}$ andel)

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Dependent Types

- Make pf a family of types indexed by formulas
  - f : Type (type of encodings of formulas)
  - e : Type (type of encodings of expressions)
  - pf : f → Type (the type of proofs indexed by formulas: it is a proof *that f is true*)
- Examples:
  - true : f
  - and : f → f → f
  - truei : pf true
  - andi : pf A → pf B → pf (and A B)
  - **andi : ΠA:f. ΠB:f. pf A → pf B → pf (and A B)**

$$\frac{A \quad B}{A \wedge B}\text{andi}$$

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Proof Checking

- Validate proof trees by type-checking them
- Given a proof tree X claiming to prove A ∧ B
- Must check X : pf (and A B)
- We use "expression tree equality", so
  - andel (andi "1+2=3" "x=y") does *not* have type pf (3=3)
  - This is already a proof system! If the proof-supplier wants to use the fact that 1+2=3 ⇔ 3=3, she can include a proof of it somewhere!
- Thus Type Checking = Proof Checking
  - And it's quite easily *decidable*!

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Parametric Judgment

- Universal Introduction Rule of Inference

$$\frac{\vdash [a/x]A \quad (a \text{ is fresh})}{\vdash \forall x.\ A}$$

- We represent bound variables in the logic using bound variables in the meta-logic
  - all : (e → f) → f
  - Example: ∀x. x=x represented as (all (λx. eq x x))
  - Note: ∀y. y=y has an α-equivalent representation
  - Substitution is done by β-reduction in meta-logic
    - **[E/x](x=x) is (λx. eq x x) E**

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Parametric ∀ Proof Rules

$$\frac{\vdash [a/x]A \quad (a \text{ is fresh})}{\vdash \forall x.\ A}$$

- Universal Introduction
  - alli: ΠA:(e → f). (Πa:e. pf (A a)) → pf (all A)

$$\frac{\vdash \forall x.\ A}{\vdash [E/x]A}$$

- Universal Elimination
  - alle: ΠA:(e → f). ΠE:e. pf (all A) → pf (A E)

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Parametric ∃ Proof Rules

$$\frac{\vdash [E/x]A}{\vdash \exists x.\ A}$$

- Existential Introduction
  - existi: ΠA:(e → f). ΠE:e. pf (A E) → pf (exists A)

$$\frac{\vdash \exists x.\ A \qquad \begin{array}{c}\vdash [a/x]A \\ \dots \\ \vdash B\end{array}}{\vdash B}$$

- Existential Elimination
  - existe: ΠA:(e → f). ΠB:f. pf (exists A) → (Πa:e. pf (A a) → pf B) → pf B

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