University of Colorado  
Department of Computer Science  
Chaotic Dynamics – CSCI 4446/5446  
Spring 2015  
Problem Set 7

Issued: 24 February 2015  
Due: 3 March 2015

Reading: Strogatz, sections 6.0-6.5; Liz’s notes on the variational equation; Parker&Chua, Appendix B.

Bibliography:

- D. Ruelle, “Strange Attractors,” The Mathematical Intelligencer, 2:126-137, 1980. There’s a closely related paper by him in the Cvitanovic collection, which is on library reserve.

Problems:

1. [math] Derive the Jacobian $D_x \vec{F}$ for the Lorenz system:

$\vec{F}(\vec{x}, a, r, b) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a(y-x) \\ rx-y-xz \\ xy-bz \end{bmatrix}$

   \begin{equation}
   \begin{bmatrix}
   \frac{\partial \dot{x}}{\partial x} = \delta_{xx} & \frac{\partial \dot{x}}{\partial y} = \delta_{yx} & \frac{\partial \dot{x}}{\partial z} = \delta_{zx} \\
   \frac{\partial \dot{y}}{\partial x} = \delta_{xy} & \frac{\partial \dot{y}}{\partial y} = \delta_{yy} & \frac{\partial \dot{y}}{\partial z} = \delta_{zy} \\
   \frac{\partial \dot{z}}{\partial x} = \delta_{xz} & \frac{\partial \dot{z}}{\partial y} = \delta_{yz} & \frac{\partial \dot{z}}{\partial z} = \delta_{zz}
   \end{bmatrix}
   \end{equation}

2. [math] Write down the associated variational system $\dot{\delta} = D_x \vec{F} \delta$. The product of the Jacobian matrix $D_x \vec{F}$ and the $n \times n$ matrix of variations $\dot{\delta}$ yields an $n \times n$ matrix of the derivatives of the variations $\dot{\delta}$. $\delta_{xy}$, for instance, is the component of the $x$-variation that comes from the previous $y$-variation, and $\dot{\delta}_{xy}$ is its rate of change with time.
3. [programming] A combination of this variational derivative and the original system derivative from PS5 can be used to integrate the \((n^2 + n)\)-dimensional variational equation

\[
\begin{bmatrix}
\dot{x} \\ \delta
\end{bmatrix} = \begin{bmatrix}
\vec{F} \\
D_x \vec{F} \cdot \delta
\end{bmatrix}
\]

from the initial condition

\[
\begin{bmatrix}
\vec{x}_0 \\
I
\end{bmatrix}
\]

with \(t = t_0\). The time evolution of the first \(n\) elements of this set of initial conditions follows the trajectory \(\phi_t(\vec{x}_0, t_0)\). The row and column sums of the matrix formed by the next \(n^2\) elements are different ways to look at the evolved versions of the initial variations: the first column sum, for example, gives the \(x\) component of the evolved variation, while the first row sum tells you what the \(x\)-piece of the original variation has grown into.

Integrate the Lorenz variational equation using RK4 (not ARK4) from the following initial conditions for 100 steps. Use \(a = 16\), \(r = 45\), \(b = 4\), and a timestep of .001. In each case, give

- the components of the evolved matrix \(\delta\)
- the evolved variations (the column sums of \(\delta\))

at the endpoint of the trajectory. Use \(t_0 = 0\) (this is really immaterial, as the system is autonomous). You need only turn in these twelve numbers for each of the following questions; no plots or intermediate values are necessary.

(a) \([x\ y\ z\ \delta_{xx}\ \delta_{xy}\ \delta_{xz}\ \delta_{yy}\ \delta_{yz}\ \delta_{zx}\ \delta_{zy}\ \delta_{zz}] = [0\ 1\ 2\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1]\)

(b) \([10\ -5\ 2\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1]\)

(c) \([0\ -1\ 2\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1]\)

(d) Look carefully at the evolved matrices of variations and describe some of their interesting features. From which point [(a), (b), or (c)] do the variations grow fastest? In which direction (\(x\), \(y\), \(z\))? Do you notice any symmetries or gross differences between the different points (e.g., “the \(y\)-variation grows really fast near point A, but less so near point B...”)?