Abstract

We present a linear time algorithm to properly color the edges of any graph of maximum degree 3 using 4 colors. Our algorithm uses a greedy approach and utilizes a new structure theorem for such graphs.
4-edge-coloring graphs of maximum degree 3 in linear time

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1 Introduction

Graph coloring is one of the most fertile and well-studied subjects in Graph Theory. An evidence of this fact may be found by browsing through the list of solved and unsolved problems in a comprehensive book [8] on graph coloring problems. The most general problem in the field is vertex coloring, since many coloring problems can be reduced to it. In the vertex coloring problem, we want to use the least number of colors to color the vertices of a graph, one color per vertex, in such a way that no two adjacent vertices are assigned the same color. This least number of colors is called the chromatic number $\chi$ of the graph. We are interested in a related edge coloring problem. In this problem, we want to use the least number of colors to color the edges of a graph, one color per edge, in such a way that no two adjacent edges are assigned the same color. This least number of colors is called the chromatic index $\chi'$ of the graph. A theorem of Vizing states that the chromatic index of a simple graph is at most one larger than its maximum degree $\Delta$ [15, 11]. Therefore the chromatic index of any simple graph is either $\Delta$ or $\Delta + 1$. Determining the true value proves to be an NP-complete problem, as shown by Holey[7]. In fact, [7] shows that the problem remains NP-complete even when restricted to cubic graphs, those graphs whose every vertex is incident with exactly three edges. As observed in the survey paper [6] on cubic graphs, cubic graphs often seem to be the simplest class of graphs for which a problem remains as difficult to solve as on a general graph. By studying the problem when restricted to cubic graphs we may gain insight into why the problem is difficult. Actually, we will work with cubic graphs and more. The class of graphs of interest to us properly contains the class of cubic graphs.

We will be concerned with graphs of maximum degree 3 from now on. By [7], the problem of determining the chromatic index of any such graph is NP-complete. How about a polynomial time algorithm to edge-color them using 4 colors? The proof of Vizing’s Theorem [3, 11, 1, 10] gives an $O(|V||E|)$ time algorithm to edge-color any arbitrary simple graph $G = (V,E)$ using $\Delta + 1$
colors. When specialized to graphs of maximum degree 3, we get an $O(|V|^2)$
algorithm to edge-color them using 4 colors. Now edge-coloring a graph $G$ is
equivalent to vertex-coloring its line graph $L(G)$. In the vertex coloring problem,
Brooks’ Theorem [2] states that a connected graph that is neither a complete
graph nor an odd cycle has chromatic number no bigger than its maximum
degree. Suppose $G = (V, E)$ has maximum degree 3. Then $L(G)$ has maximum
degree at most 4 and it has at most $(3/2)|V|$ vertices. By Brooks’ Theorem
$L(G)$ can be vertex-colored with 4 colors. Thus $G$ can be 4-edge-colored. The
proof of Brooks’ theorem when specialized to graphs of maximum degree 4 gives
an $O(|V|)$ time algorithm for 4-vertex-coloring them. However, the algorithm
resulting from the proof [1, 10] has several drawbacks. It is complicated; it
requires computation of 2-connectivity [13, 4] and triconnected components [14];
and it requires extra housekeeping. In this paper we develop a direct $O(|V|)$
time algorithm to 4-edge-color any graph of maximum degree 3. Our algorithm
is very simple. It uses depth-first-search [13, 4] and the greedy approach. Its
simplicity rests on the fact that we may decompose any such graph into two
distinct parts: a forest and a collection of vertex-disjoint cycles. It does not de-
depend on the truth of either Brooks’ or Vizing’s Theorem. It works equally well
on both simple graphs and graphs that have multiple edges (but no self-loops).
Thus it algorithmically shows that any (multi)graph of maximum degree 3 can
be 4-edge-colored. In an early paper [9] Johnson gives a proof that any cubic
graph can be 4-edge-colored. The algorithm that results from his proof requires
dynamic maintenance of 2-edge-connectivity under contraction of vertices and
addition of edges. This latter problem is not known to have a linear time algo-
rithm. Another proof that 4 colors suffice to properly color the edges of planar
cubic graphs is given by Golovina & Yaglom [5, 12]. Their inductive proof
Can be turned into a linear time algorithm. The disadvantages of the resulting
algorithm are that it is complex and requires recoloring.

The rest of this paper is organized as follows. Section 2 defines relevant
terms to be used. Section 3 concerns the decomposition theorem. Section 4
describes the coloring algorithm. Section 5 concludes our paper. Appendix A
gives detailed pseudocode for a decomposition algorithm. Appendix B gives
detailed pseudocode for the coloring algorithm.

2 Terminology

Definition 1. Let $G = (V, E)$ be a finite graph with vertex set $V$ and edge
set $E$. We allow $G$ to have multiple edges but no self-loops. We use $n$ to denote
$|V|$ and $m$ to denote $|E|$. A graph $H = (V', E')$ is a subgraph of $G = (V, E)$
if $V' \subseteq V$ and $E' \subseteq E$. A cycle is a connected graph whose every vertex has
degree 2. Two graphs are edge-disjoint if they have no common edge; they are
vertex-disjoint if they have no common vertex. If $v$ is a vertex, we write $d(v)$
for its degree. A forest is a graph without cycles. A tree is a connected forest.
A node is a vertex in a forest. A leaf is a node of degree 1. A cycle is even if it
has an even number of edges. If $k$ is a positive integer let $\{1, 2, \ldots, k\}$ be a set

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of colors. A (proper) $k$-edge-coloring of $G$ is an assignment of some color from the color set $\{1, 2, \ldots, k\}$ to each edge of $G$ in such a way that no two adjacent edges are assigned the same color.

Any graph-theoretic terms we use before defining is standard and its meaning can be found in [1, 11, 10].

3 Decomposition Theorem

**Theorem 1.** Let $G$ be a graph of maximum degree 3. Then $G$ can be decomposed into 2 edge-disjoint subgraphs $C$ and $F$, where $C$ is a collection of vertex-disjoint cycles and $F$ is a forest of maximum degree no bigger than 3.

**Proof.** Suppose $G$ contains a cycle $C_1$. Remove all edges of $C_1$ from $G$. If the resulting graph still contains cycles, keep removing them. Say we have removed the cycles $C = \{C_1, C_2, \ldots, C_p\}$ until the resulting graph $F$ contains no cycles (i.e., $F$ is a forest). Any two distinct cycles $C_i, C_j$ must be vertex-disjoint. To see this, suppose otherwise that $v$ is a vertex on both $C_i$ and $C_j$. Since $C_i$ and $C_j$ are edge-disjoint, $v$ is incident with 2 edges of $C_i$ and 2 edges of $C_j$. Thus $d(v) \geq 4$. This contradicts the assumption $d(v) \leq 3$. Being a subgraph of $G$, the maximum degree of $F$ cannot exceed 3. $\square$

Note that decomposition of $G$ into forest and cycles is not unique. The theorem implies that the edges of $G$ can be partitioned into two kinds: cycle edges and tree edges. Figure 1 shows a decomposed cubic graph. Dashed arcs are cycle edges; solid lines are tree edges. Appendix A describes a decomposition algorithm in detail.

4 Coloring Algorithm

Let $G = (V, E)$ be an input graph of maximum degree 3. Wlog assume $G$ is connected. The coloring algorithm works in two stages. The first stage is
to decompose $G$ into forest $F$ and cycles $C_1, \ldots, C_p$. The actual coloring is done in the second stage. All the trees are colored first; the cycles are colored later. The coloring is done using a greedy approach. It systematically picks an edge and assigns the least available color from the color set $\{1, 2, 3, 4\}$ to that edge. For each tree $T$ in $F$, it does a depth-first search starting from some arbitrary vertex in $T$. Tree edges are assigned colors in the order that they are discovered by the search. For each cycle $C_i$ the algorithm tours $C_i$ and assigns colors to the edges in the order that they are discovered, taking care to start the tour from an appropriate vertex and go around $C_i$ in an appropriate direction. It will be convenient to define some more terms. A vertex $v$ on $C_i$ is pure if its degree in $G$ is 2. A vertex $v$ on $C_i$ is tainted with $c$ if the tree edge incident with $v$ has been assigned color $c$. We also say that $C_i$ is $d$- tainted if $|\{c \in \{1, 2, 3, 4\} : \text{some } v \text{ on } C_i \text{ is tainted with } c\}| = d$. The algorithm first determines if $C_i$ contains any pure vertex. If a pure vertex $w$ is found, let $uv$ be a cycle edge. It starts the tour from $v$ but away from $w$. If no vertex on $C_i$ is pure, it counts the number of colors that $C_i$ is tainted with. Three cases are distinguished whose proof will be given later.

Case 1: $C_i$ is 1-tainted. It starts the tour from any vertex and can go around $C_i$ in any direction.

Case 2: $C_i$ is 2-tainted. It tries to find two consecutive vertices $w, v$ that are both tainted with the same color.

Subcase 2.1: Such a pair of vertices exists. It tours $C_i$ starting from $v$ but away from $w$.

Subcase 2.2: Such a pair of vertices does not exist. It tours $C_i$ starting from any vertex and can go around $C_i$ in any direction.

Case 3: $C_i$ is 3-tainted. It finds three consecutive vertices $w, u, v$ no two of which are tainted with the same color. It then starts the tour from $u$ and can go around $C_i$ in any direction. See Appendix B for detailed pseudocode.

Example 1. Figure 2 shows how the algorithm works on non-pure cycles of each type. A number at each vertex is the color it is tainted with. A number on each edge is the color assigned by the algorithm. In the first three examples the algorithm tours the cycle starting with edge $ux$ and ending with edge $uw$. So $ux$ is the first and $uw$ the last edge to be assigned color. In the fourth example the algorithm tours the cycle starting with edge $vu$ and ending with edge $uw$. So $vu$ is the first and $uw$ the last edge to be assigned color.

The first cycle is tainted with one color 3. The second cycle is tainted with two colors $\{1, 2\}$, and it has consecutive vertices $u, v$ both tainted with color 1. The third cycle is tainted with two colors $\{2, 3\}$ and every edge on it joins a vertex tainted with 2 to a vertex tainted with 3. The fourth cycle is tainted with three colors $\{1, 2, 3\}$. Vertices $w, u$ and $v$ are consecutive and no two of them are tainted with the same color.

Proof of Correctness. Depth-first search on the trees using greedy color choice on entrance works since each vertex of the tree has at most 3 incident edges, so our algorithm never gets stuck since it has 4 potential colors available. In fact, it never uses color 4. So the leaves are tainted with 1, 2, or 3 only.
Figure 2: Examples showing how the algorithm works on each type of cycles

Note that the only time that greedy color choice on a cycle can fail is on the very last edge. So it suffices to show that when our algorithm is about to assign color to the last edge $e$ in its tour of a cycle $C_i$, no more than 3 colors had been used to color the (at most) 4 edges adjacent to $e$. This clearly holds if $C_i$ contains a pure vertex. Otherwise, consider each case separately.

Case 1: $C_i$ is 1-tainted. Suppose $uv$ is the last edge in the tour of $C_i$. Since $u$ and $v$ are tainted with the same color, when our algorithm is about to color edge $uv$, no more than 3 colors had been used to color the 4 edges adjacent to $uv$.

Case 2: $C_i$ is 2-tainted. First suppose $C_i$ contains consecutive vertices $w, v$ tainted with the same color. Edge $uv$ is the last edge to be assigned color but at most 3 colors had been used to color the 4 edges adjacent to $uv$. Next suppose every 2 consecutive vertices of $C_i$ are tainted with different colors. Then $C_i$ is even; and thus the first edge and the penultimate edge in the tour of $C_i$ had been assigned the same color. Therefore, exactly 3 colors had been used to color the 4 edges incident to $uv$.

Case 3: $C_i$ is 3-tainted with $\{1, 2, 3\}$. Then we can find consecutive vertices $v, u, w$ such that no two of them are tainted with the same color; for otherwise $C_i$ would be at most 2-tainted. If our algorithm picks color $c$ for the first edge $uv$, then $c$ is the same color that $w$ is tainted with! So when it is about to color the last edge $vw$, no more than 3 colors had been used to color the 4 edges adjacent to $uv$.

\[ \square \]

5 Conclusion

We have presented a simple method to linearly 4-edge-color any graph of maximum degree 3. Our approach relies on a new decomposition theorem for such a graph. It is entertaining to ask whether this approach can be extended to attack similar problems.
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6 Appendix

A Pseudocode for a Decomposition Algorithm

The proof of the decomposition theorem suggests the following high-level algorithm to decompose $G$ into cycles $C$ and forest $F$. This algorithm uses path-based depth-first search approach [4]. It maintains a graph $F$ that is $G$ with some cycles deleted. It also maintains a path $P$ in $F$. Initially $F$ is the given graph $G$ and $C = \emptyset$. It returns the collection $C$ of disjoint cycles, and the remaining graph as forest $F$. Each cycle in $C$ is a sequence of consecutive vertices on it. The high-level algorithm is as follows.

If all vertices of $F$ have been discovered stop. Otherwise start a new path $P$ by choosing a vertex $v$, marking $v$ discovered, and setting $P = \langle v \rangle$. Continue growing $P$ as follows.

To grow a path $P = \langle v_1, \ldots, v_k \rangle$ choose an edge $v_kw$ such that $w \neq v_{k-1}$ and do the followings:

- If $w$ has not been discovered, mark $w$ as discovered, add it to $P$, making it the last vertex of $P$. Continue growing $P$.
- If $w \in P$, say $w = v_i$, add a new cycle $\langle v_i, v_{i+1}, \ldots, v_k \rangle$ to $C$. Delete every edge on this cycle from $F$, and delete every vertex on this cycle from $P$. If $P$ is now nonempty continue growing $P$. Otherwise try to start a new path $P$.
- If every edge $v_kw \neq v_kv_{k-1}$ has $w \notin P$ and $w$ already discovered, then $v_k$ cannot be on any cycle in $F$. Delete $v_k$ from $P$. If $P$ is now nonempty continue growing $P$. Otherwise try to start a new path $P$.

To prove that the algorithm is correct, we need to show that each sequence of vertices added to $C$ is indeed a cycle in $G$ and that the remaining graph $F$ is acyclic. We leave the proof to the reader.

Now we give an implementation that achieves linear time. Number the vertices of $G$ by consecutive integers from 1 to $n$. We use two data structures, an array and an array-stack. An array-stack is one that can be accessed/updated either as a regular array or as a stack. Array-stack $S$ is used to represent the path $P$. An array $I[1..n]$ is used to store stack indices. It is also used to indicate various status of a vertex. More precisely for a given vertex $v$ at any
point in time,

\[ I[v] = \begin{cases} 
0 & \text{if } v \text{ has never been in } P; \\
 n + 1 & \text{if } v \text{ is known not to be on any cycle; } \\
 n + 1 + p & \text{if } v \text{ is known to be a vertex on the cycle } C_p; \\
 j & \text{if either } v \text{ is currently in } P \text{ and } S[j] = v; \text{ or } v \text{ is not} \\
 & \text{currently in } P \text{ but } v \text{ is known to be on the same cycle} \\
& \text{as } S[j], S[j + 1], \ldots, S[\text{TOP}(S)]; 
\end{cases} \]

We use the following operations to manipulate a stack \( S \): \text{PUSH}(x, S) \) adds \( x \) to \( S \) at the new top of \( S \). \text{POP}(S) \) removes the value at the top of the stack and returns that value. \text{TOP}(S) \) is the index of the value at the top of the stack. Hence \( S[\text{TOP}(S)] \) is the value at the top of the stack.

A variable \( p \) is used to keep track of the cycle number. The algorithm consists of a main routine \text{decompose()} \) and a recursive procedure \text{cycle()} \).

\begin{verbatim}
procedure decompose(G) {
    empty stack S;
    for v \in V do I[v] \leftarrow 0;
    p \leftarrow 0;
    for v \in V do
        if I[v] = 0 then cycle(v);
}

procedure cycle(v) {
    \text{PUSH}(v, S); I[v] \leftarrow \text{TOP}(S); /* add } v \text{ to end of } P */
    for each edge vw do 
        if I[w] = 0 then 
            cycle(w);
            if I[w] < I[v] then 
                \{ I[v] \leftarrow I[w]; I[w] \leftarrow n + 1 + p; \text{POP}(S); return; \}
            else if I[w] = I[v] then 
                \{ I[v] \leftarrow I[w] \leftarrow n + 1 + p; \text{POP}(S); return; \}
            else if I[w] < I[v] - 1 then 
                /* a new cycle is found */
                increase p by 1;
                C_p \leftarrow (S[I[w]], S[I[w] + 1], \ldots, S[I[v] - 1], S[I[v]]);
                I[v] \leftarrow I[w]; \text{POP}(S); return; 
        \}
    I[v] \leftarrow n + 1; \text{POP}(S);
}
\end{verbatim}

Proving that the above pseudocode correctly implements the high-level algorithm is again left to the reader. Our input graph has \( m \leq (3/2)n \). Therefore, the above implementation takes \( O(n) \) time since it spends \( O(1) \) time on each vertex and each edge.

**B  Pseudocode for the Coloring Algorithm**

We now give detailed pseudocode for the coloring algorithm \text{greedy()}.

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procedure greedy(G = (V, E)) {  
  for each e ∈ E do  
    color[e] ← 0;  
  decompose G into cycles {C₁, ..., Cₚ} and forest F = (V, E');  
  for each v ∈ V do  
    discovered[v] ← false;  
  for each v ∈ V do  
    if not discovered[v] then dfs(v);  
  for i ← 1 to p do  
    if Cᵢ contains a pure vertex w then {  
      let w, v, x be consecutive vertices on Cᵢ;  
      tour(v, x, Cᵢ);  
    } else if Cᵢ is 1-tainted then {  
      let v, x be consecutive vertices on Cᵢ;  
      tour(v, x, Cᵢ);  
    } else if Cᵢ is 2-tainted then {  
      let w, v be consecutive vertices on Cᵢ that are both 
      tainted with the same color if such a pair w,v exists, 
      otherwise let w, v be any consecutive vertices on Cᵢ;  
      let x be such that vx is an edge on Cᵢ and x ≠ w;  
      tour(v, x, Cᵢ);  
    } else { /* Cᵢ is 3-tainted */  
      let w,v,u be consecutive vertices on Cᵢ such that 
      no two of them are tainted with the same color;  
      tour(v, u, Cᵢ);  
    }  
}  
procedure dfs(v) {  
  discovered[v] ← true;  
  for each edge vw ∈ E' do  
    if not discovered[w] then {  
      color[vw] ← least_color(v, w);  
      dfs(w);  
    }  
}  
procedure tour(v, w, C) {  
  s ← v;  
  color[vw] ← least_color(v, w);  
  while w ≠ s do {  
    x ← vertex on C adjacent to w but distinct from v;  
    color[vx] ← least_color(w, x);  
    v ← w; w ← x;  
  }  
}  
function least_color(v, w) {  
  for e ← 1 to 4 do  
    if c ≠ color[e] for all 4 edges e ∈ E adjacent to vw then 
      return c;  
  }
Our algorithm takes $O(n)$ time to run since it spends $O(1)$ time on each vertex and each edge.

References