Model Completeness of an Algebra of Languages *

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Abstract

An algebra $<L, f, g>$ of languages over a finite alphabet $\Sigma = \{a_1, \ldots, a_n\}$ is defined with operations $f(L_1, \ldots, L_n) = a_1 L_1 \cup \ldots \cup a_n L_n \cup \lambda$ and $g(L_1, \ldots, L_n) = a_1 L_1 \cup \ldots \cup a_n L_n$ and its first order theory is shown to be model complete. A characterization of the regular languages as unique solutions of sets of equations in $<L, f, g>$ is given and it is shown that the subalgebra $<R, f, g>$ where $R$ is the set of regular languages is a prime model for the theory of $<L, f, g>$. 
Let $\Sigma = \{a_1, \ldots, a_n\}$ be a finite alphabet and $\Sigma^*$ the free semigroup with empty word $\lambda$ generated by $\Sigma$. Let $L$ be the class of all languages over $\Sigma$, i.e., all subsets of $\Sigma^*$. We introduce two $n$-ary operations on the languages of $L$:

$$f(L_1, \ldots, L_n) = a_1 L_1 u \ldots u a_n L_n u\{\lambda\}$$

$$g(L_1, \ldots, L_n) = a_1 L_1 u \ldots u a_n L_n$$

where $a_i L_i$ denotes the language obtained by prefixing all the words of $L_i$ with the letter $a_i$.

Our first result is the following theorem which follows from Theorem 3 of Mycielski and Perlmutter [3]:

**Theorem 1:** The first order theory of the algebra $<L, f, g>$ is model complete.

**Proof:** Let us define a simple bijection between $L$ and the set of infinite, oriented trees with nodes labeled from $\{f, g\}$, each node having $n$ successors. Given such a tree, label the edges emanating from each node with the letters $a_1$ through $a_n$ from left to right. Associate with the tree the language consisting of all words of $\Sigma^*$ corresponding to the consecutive labels of the edges of any path leading from the root to a node labeled $f$. It follows that the algebra $<L, f, g>$ is isomorphic to the algebra $R_\sigma$ of [3], where $\sigma$ specifies the two $n$-ary function symbols $f$ and $g$. Thus by Theorem 3 of [3], $<L, f, g>$ is model complete.

We now consider sets of equations for $<L, f, g>$, i.e., sets of equations written solely in terms of the function symbols $f$ and $g$ and variables $x_i$. 

Let us say that a language \( L \) is uniquely determined by a set of equations \( E \) and a variable \( x_p \) iff \( E \) is satisfiable in \( \langle L, f, g \rangle \) and every assignment to the variables of \( E \) which satisfies \( E \) assigns \( L \) to \( x_p \).

**Lemma 1:** If \( L \) is uniquely determined by some set of equations and a variable, then \( L \) is uniquely determined by a set of equations \( E \) and \( x_1 \), where \( E \) has the unknowns \( x_1, \ldots, x_m \) and is of the form \( \{ x_i = t_i : 1 \leq i \leq m \} \), the \( t_i \)'s being terms which are not variables.

**Proof:** Assume that \( L \) is uniquely determined by the set of equations \( D \) and the variable \( x_p \). We define an equivalence relation, \( \equiv \), on the variables appearing in \( D \) by

\[
x_i \equiv x_j \iff D \Rightarrow x_i = x_j \text{ is true in } \langle L, f, g \rangle.
\]

From each equivalence class, we choose a representative, insuring that \( x_p \) is chosen as a representative of its class. We then replace all the variables in \( D \) by their representatives, obtaining a set of equations \( D' \) which is equivalent to \( D \) with respect to the remaining variables.

Using the isomorphism from Theorem 1 and Lemma B, Case I from [3], we convert \( D' \) to an equivalent set of equations \( D'' = \{ x_i = t_k : 1 \leq k \leq r \} \) where the \( x_i \)'s are distinct variables and the \( t_k \)'s are terms which are not variables. Now notice that the system \( D'' \) is satisfiable in \( \langle L, f, g \rangle \) for every assignment of the variables which do not occur on the left-hand side of any equation of \( D'' \) (see [3], formula (2)). Hence \( x_p \) appears on the left-hand side of some equation in \( D'' \). To finish the proof, we substitute every variable of \( D'' \) which does not appear on the left-hand side of any equation by the variable \( x_p \). Finally, we rename the variables to obtain a set of equations \( E \) of the desired form.
Our second theorem provides a characterization of the class of regular languages (see e.g. [2]) in terms of sets of equations in \(<L,f,g>\).

Theorem 2: The following are equivalent:

(i) \(L\) is uniquely determined by some set of equations and a variable in \(<L,f,g>\),

(ii) \(L\) is uniquely determined by a set of equations \(E\) in unknowns \(x_1, \ldots, x_m\) and the variable \(x_i\), where \(E\) is of the form \(\{x_i = \phi_i(x_{i_1}, \ldots, x_{i_n}) : 1 \leq i \leq m \}\) and \(\phi_i \in \{f,g\}\) for each \(i\),

(iii) \(L\) is regular.

Proof: We first show (i) \(\Rightarrow\) (ii). By Lemma 1, we may assume that \(L\) is uniquely determined by \(E_0 = \{x_i = t_i : 1 \leq i \leq m \}\) and the variable \(x_i\) where \(E_0\) has the properties stated in the lemma. From \(E_0\) we will produce a set of equations \(E\) of the form specified in (ii) in the following way. Initially let \(E = E_0\). Then, given any equation of \(E\) of the form \(x_j = \phi(u_1, \ldots, u_n)\) where the \(u_i\)'s are terms and for some \(k: 1 \leq k \leq n, u_k\) is not a variable, replace this equation with the two equations \(x_j = \phi(u_1, \ldots, u_{k-1}, x_i, u_{k+1}, \ldots, u_n)\) and \(x_i = u_k\) where \(i\) is the least integer such that \(x_i\) does not appear in any equations of \(E\) up to this point. We continue this operation as long as feasible. Since terms are of finite depth, this process terminates and it is apparent that it produces a set of equations \(E\) of the required form which is equivalent to \(E\) with respect to the original variables.
To show (ii) \(\Rightarrow\) (iii), we transform \(E\) into a finite automaton
\[ M = \langle Q, \Sigma, \delta, x_1, F \rangle \]
accepting precisely the language \(L\). \(Q\), the set of states of \(M\), is defined to be the set of variables of \(E\).
\(E = \{a_1, \ldots, a_n\}\) is the alphabet of \(M\). \(\delta\), the transition function, is defined by 
\[ \delta(x_k, a_j) = x_{i_{j}} \]
iff \(E\) has an equation of the form
\[ x_k = \phi(x_{i_1}, \ldots, x_{i_n}) \]
\(x_1\) is the start state and \(F\), the set of accepting states, is the set of those variables \(x_k\) for which an equation of the form
\[ x_k = f(x_{i_1}, \ldots, x_{i_n}) \]
is in \(E\). In view of the definition of the operations \(f\) and \(g\), it is obvious that \(M\) must accept \(L\), hence \(L\) is regular.

To see that (iii) \(\Rightarrow\) (i) it suffices to observe that given any deterministic finite automaton 
\[ M = \langle Q, \Sigma, \delta, x_1, F \rangle \]
with \(Q \subset \{x_1, x_2, \ldots\}\) and \(\Sigma = \{a_1, \ldots, a_n\}\) we can easily reverse the above construction, obtaining a set of equations \(E\) such that \(E\) and \(x_1\) uniquely determine the language \(L\) accepted by \(M\).

Let \(R\) be the class of regular languages over \(E\). Since \(R\) is closed under the operations \(f\) and \(g\), \(\langle R, f, g \rangle\) is a subalgebra of \(\langle L, f, g \rangle\). From Theorem 2 we may easily deduce the following corollary:

Corollary 1: Every finite set of equations in \(f\) and \(g\) which has a solution in \(\langle L, f, g \rangle\) has a solution in \(\langle R, f, g \rangle\).

However, using [3] again, we obtain the following stronger result:
Theorem 3: \( <R,f,g> \) is an elementary subalgebra of \( <L,f,g> \) and is a prime model for its theory.

Proof: By the isomorphism of the proof of Theorem 1, \( <R,f,g> \) is isomorphic to the algebra \( A_{\sigma} \) of [3] where \( \sigma \) is as before. Our result follows from part (ii) of Theorem 3 of [3]. \( \square \)

In closing, let us mention a few open problems.

1. Is the theory of \( <L,f,g> \) decidable?

2. In [1], J. H. Conway defines and studies the operations:

\[
\frac{\partial}{\partial a_i}(L) = \{ w : a_i \in w \}
\]

which are existentially first order definable from \( f \) and \( g \). Does the theory of the algebra \( <L,f,g,\frac{\partial}{\partial a_1}, ..., \frac{\partial}{\partial a_n}> \) admit elimination of quantifiers?

References


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