ON k-STABLE FUNCTIONS

by

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Abstract

We prove that a $k$-continuous or a $k$-stable function cannot depend on more than $k^4 k^{-1}$ variables and related facts.

A function $f: \{0,1\}^n \to \mathbb{R}$, where $\mathbb{R}$ is any set is called $k$-continuous iff for every $x = (x_1,\ldots,x_n) \in \{0,1\}^n$ there exists a sequence $1 \leq i_1 < \ldots < i_p \leq n$, where $p \leq k$, such that for every $y = (y_1,\ldots,y_n) \in \{0,1\}^n$ if $(y_{i_1},\ldots,y_{i_p}) = (x_{i_1},\ldots,x_{i_p})$ then $f(y) = f(x)$. This property was studied in [1,2,3,5].

Now we will study a larger class of functions $f: \{0,1\}^n \to \mathbb{R}$ called $k$-stable. To explain this property, for every $x = (x_1,\ldots,x_n) \in \{0,1\}^n$ and every $i$ with $1 \leq i \leq n$ we put

$$x^i = (x_1,\ldots,x_{i-1},1-x_i,x_{i+1},\ldots,x_n).$$

Now $f$ is called $k$-stable iff for every $x = (x_1,\ldots,x_n) \in \{0,1\}^n$ there exist $1 \leq i_1 < \ldots < i_p \leq n$, where $p \leq k$, such that for every $i \notin \{i_1,\ldots,i_p\}$, $1 \leq i \leq n$, we have $f(x^i) = f(x)$. Thus, of course, $k$-continuity implies $k$-stability.

Examples. 1. The function $f: \{0,1\}^4 \to \{0,1\}$ defined by $f(x) = 0$ if $x \in \{(0,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1)\}$, and $f(x) = 1$ otherwise, is 2-continuous.

2. The function $f: \{0,1\}^{10} \to \{0,1\}$ defined by

$$f(x_1,\ldots,x_{10}) = x_1 \text{ if } x_1 = x_2, \quad f(x_1,\ldots,x_{10}) = 0 \text{ if } x_1 = x_3 = x_4 = 0 \text{ or } x_1 = x_5 = x_6 = 0 \text{ or } x_2 = x_7 = x_8 = 0 \text{ or } x_2 = x_9 = x_{10} = 0,$$

and
f(x_1, \ldots, x_{10}) = 1 \text{ otherwise, is } 3\text{-continuous (see fig. 1). For other examples of } k\text{-continuous Boolean functions see [2], Notes 3, 4, and 5, and [3].}

3. The function \( f: \{0,1\}^4 \rightarrow \{0,1\} \) defined by \( f(x) = 0 \) if \( x \in \{(0,0,0,0), (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)\} \), and \( f(x) = 1 \) otherwise, is 2-stable but not 2-continuous. (see fig. 2)

\begin{center}
\text{Fig. 1}
\end{center}

\begin{center}
\text{Fig. 2}
\end{center}

A function \( f: \{0,1\}^n \rightarrow R \) is said to \underline{depend} on the variable \( x_i \) iff there exists a sequence \( y = (y_1, \ldots, y_n) \in \{0,1\}^n \) such that \( f(y) \neq f(y^i) \). And a function \( f: \{0,1\}^n \rightarrow R \) is called \underline{Boolean} iff \( R \subseteq \{0,1\} \).

E.g.: the functions of Examples 1 and 3 depend on 4 variables and the function of Example 2 depends on 10 variables, and all are Boolean.

In [2] we have studied the maximum number of variables on which a \( k\text{-continuous Boolean function can depend. It turns out that such} \)
a maximum exists and we will denote it here (unlike in [2]) by $\varphi_2(k)$.

The following problem is still unsolved

(P₁) Does there exist for every $n < \varphi_2(k)$ a $k$-continuous Boolean function which depends just on $n$ variables?

It is not hard to prove that $\varphi_2(1) = 1$ and $\varphi_2(2) = 4$ (see Example 1). By Example 2 we have $\varphi_2(3) \geq 10$. It seems that

$\varphi_2(3) = 10$.

We shall also study functions $f : X \to \mathbb{R}$, where $X$ can be a proper subset of $\{0,1\}^n$. We shall say that $f$ is total if $X = \{0,1\}^n$ and partial if $X \neq \{0,1\}^n$. For a partial $f$ we shall say that $f$ depends on the variable $x_i$ if there exists a $y \in X$ such that $y^i \in X$ and $f(y) \neq f(y^i)$. Also $f$ is called Boolean if $R \subseteq \{0,1\}$.

It is called $k$-continuous if for every $x \in X$ there exists $1 \leq i_1 < \ldots < i_p \leq n$ such that $p \leq k$ and for every $y \in \{0,1\}^n$ if $(y_{i_1}, \ldots, y_{i_p}) = (x_{i_1}, \ldots, x_{i_p})$ then $y \in X$ and $f(y) = f(x)$.

(In [2] this property was called regular $k$-continuity.) $f$ is called $k$-stable if for every $x \in X$ there exists $1 \leq i_1 < \ldots < i_p \leq n$ such that $p \leq k$ and for all $i \notin \{i_1, \ldots, i_p\}$, $1 \leq i \leq n$, we have $x_i \in X$ and $f(x^i) = f(x)$.

(P₂) For which $k, n, \ell$ is it true that $k$-stability of $f : \{0,1\}^n \to \{0,1\}$ implies $\ell$-continuity of $f$? (For $k = \ell = n-1$ it is so.)

(P₃) What is the maximum height (see [3]) of a total $k$-stable function? (The maximum height of a total $k$-continuous function is $k^2$ as proven in [3].)

Let now $\varphi(k)$, $\varphi^*(k)$, or $\varphi_2^*(k)$, denote the maximum number
of variables on which a k-continuous function which is total, partial or partial Boolean, respectively, can depend. Also let \( \psi(k) \), or \( \psi^*(k) \), denote the maximum number of variables on which a k-stable function which is total, or partial, respectively, can depend.

We shall prove that all these maxima exist. We have of course

\[
\varphi_2(k) \leq \varphi_2^*(k) \leq \varphi_1^*(k) \leq \psi^*(k),
\]

\[
\varphi_2(k) \leq \varphi(k) \leq \psi(k) \leq \psi^*(k) \text{ and } \varphi(k) \leq \varphi^*(k).
\]

The main result of this paper is that \( \psi^*(k) \leq k^{4k-1} \).

(P₄) Is any of the above inequalities sharp for large enough \( k \)?

In [2] (Theorem 17A and Note 6) we have proven that

\[
2(k-2) + 4\binom{2(k-2)}{k-2} \leq \varphi_2(k) \leq \varphi_2^*(k) \leq (2k-1)\binom{2(k-1)}{k-1},
\]

and we gave (Theorem 23) a different combinatorial interpretation of the quantity \( \varphi_2^*(k) \) (see also [4]). Again it is easy to prove that \( \varphi_2^*(1) = 1 \) and \( \varphi_2^*(2) = 4 \) and it seems that \( \varphi_2^*(3) = 10 \). The analogs of problem (P₁) for \( \varphi_2^*, \varphi, \varphi^*, \psi \) and \( \psi^* \) are also open.

Now we will prove that \( \psi^*(1) = 1 \). (Concerning \( \psi(2) \) and \( \psi^*(2) \) we know only that \( 4 \leq \psi(2) \leq \psi^*(2) \leq 8 \) (by Example 3 and the general fact \( \psi^*(k) \leq k^{4k-1} \) proved below)). First we need an auxiliary proposition. Let \( I \) be the interval \([0,1]\).

**Proposition.** If \( H \) is a nonempty set of edges of the \( n \)-cube \( I^n \) such that every vertex of the graph \( H \) has valency not less than \( n-1 \), then either the union \( \bigcup H \) is connected or \( H \) consists of all the edges of two disjoint \((n-1)\)-faces of \( I^n \).
Proof. We proceed by induction on \( n \). For \( n = 1 \) the Proposition is obvious. Suppose that it is true for \( n - 1 \). If \( U \cap H \) is connected we are done, thus suppose that it is disconnected. Let \( F_0 \) and \( F_1 \) be two disjoint \((n-1)\)-faces of \( I^n \). Let \( A_0 = F_0 \cap U \cap H \) and \( A_1 = F_1 \cap U \cap H \). If \( A_0 \) is connected and \( A_1 \) is connected then all vertices of \( A_0 \) which are of valency \( n-2 \) must be connected in \( U \cap H \) to some vertices in \( F_1 \). Those are in \( A_1 \) and hence \( U \cap H \) is connected contrary to our assumption. Thus \( A_0 \) has no vertices of valency \( n-2 \) and hence it is the union of all the edges of \( F_0 \). Similarly \( A_1 \) must be the union of all the edges of \( F_1 \) and the conclusion of the Proposition follows. Now suppose that \( A_0 \) is connected but \( A_1 \) is not. Then, by the inductive assumption, \( A_1 \) is a union of all the edges of two disjoint \((n-2)\)-faces of \( F_1 \). Then every vertex of \( A_1 \) must be connected in \( U \cap H \) to a vertex of \( F_0 \). It follows that \( U \cap H \) is connected, contrary to our assumption. By symmetry, there remains only the case when both \( A_0 \) and \( A_1 \) are disconnected. Then, by the inductive assumption both are unions of all the edges of two disjoint \((n-2)\)-faces of \( F_0 \) and \( F_1 \) respectively and every edge from \( F_0 \) to \( F_1 \) is in \( H \). Thus again \( H \) consists of all the edges of two disjoint \((n-1)\)-faces of \( I^n \).

Remark: Recently James Fickett refined the above Proposition proving that if all vertices of \( U \cap H \) have at least \( n-k \) edges then \( U \cap H \) has at least \( 2^{n-k} \) vertices and hence at most \( 2^k \) connected components and related results (to appear).

Corollary. \( f^*(1) = 1 \), i.e., a 1-stable function \( f: X \to R \) depends on one variable at most.
Proof. If \( f \) is a constant function the conclusion is trivially true. Thus let us assume that \( u \) and \( v \) are two different values of \( f \). Let \( H_0 \) be the set of all edges of \( I^m \) with both vertices in \( f^{-1}(u) \) and \( H_1 \) the set of all edges of \( I^m \) with both vertices in \( f^{-1}(v) \). Then let \( H = H_0 \cup H_1 \). Of course \( \cup H \) is disconnected. Since \( f \) is \( 1 \)-stable \( H \) satisfies the assumption of Proposition, and the Corollary follows from the Proposition.

Now we shall prove the main result of this paper.

Theorem 1. \( \psi^*(k) \leq k^4k^{-1} \).

Proof. Let \( X \subseteq \{0,1\}^n \) and \( f: X \rightarrow R \) be \( k \)-stable. For each \( i, 1 \leq i \leq n \), we put

\[
A_i = \{ x \in X: x^i \in X \text{ and } f(x^i) \neq f(x) \},
\]

and, for \( j \neq i, 1 \leq j \leq n \) and \( b \in \{0,1\} \),

\[
A_{i,j}b = \{ x \in A_i: x^j = b \}.
\]

We shall prove by induction on \( n \) the following lemma.

\((L_1)\) If \( n \geq 2k \) and \( |A_i| > 0 \) then \( |A_i| > 2^{n-2k+2} \).

Step I. \( n = 2k \). Let \( x \in A_i \). Since \( f \) is \( k \)-stable there exist \( 1 \leq i_1 < \ldots < i_k \leq n \) such that \( x^j \in X \) and \( f(x^i) = f(x) \) for every \( j \notin \{i_1, \ldots, i_k\} \). Hence \( i \in \{i_1, \ldots, i_k\} \). Also there exist \( 1 \leq j_1 < \ldots < j_k \leq n \) such that \( (x^i)^j \in X \) and \( f((x^i)^j) = f(x^i) \) for every \( j \notin \{j_1, \ldots, i_k\} \). Hence \( i \in \{j_1, \ldots, j_k\} \). Thus

\[
|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| < 2k,
\]

and, since \( n \geq 2k \), there exists some \( s \notin \{i_1, \ldots, i_k, j_1, \ldots, j_k\}, 1 \leq s \leq n \). Hence \( x, x^i, x^s, (x^i)^s \in A_i \).
and $|A_i| \geq 4$ follows.

**Step II.** $n > 2k$ and $(L_1)$ is valid for $n-1$. Choose $s$ as in the proof of Step I. Then $|A_i \cap A_{isb}| > 0$ for $b = 0,1$. Hence, by the inductive supposition, $|A_i \cap A_{isb}| \geq 2^{n-1-2k+2}$ for $b = 0,1$. Therefore, since $A_{is0} \cap A_{is1} = \emptyset$, we have $|A_i| \geq 2^{n-2k+2}$ as required in $(L_1)$.

Now we can conclude the proof of Theorem 1. By the Corollary we can assume without loss of generality that $k > 1$ and also that $f$ depends on all its $n$ variables and $n \geq 2k$. Let $p_i$ be the probability that $x \in A_i$, $x$ being uniformly distributed over $X$, i.e., $p_i = |A_i|/|X|$. Since $f$ depends on $n$ variables $|A_i| > 0$ for all $i$. Hence, by $(L_1)$, we have $|A_i| \geq 2^{n-2k+2}$. Since $|X| \leq 2^n$ we get

(1) \hspace{1cm} p_i \geq 4^{k+1}.

Notice that

(2) \hspace{1cm} \sum_{i=1}^{n} p_i = \frac{1}{|X|} \sum_{x \in X} |\{i : x \in A_i\}|,

and, since $f$ is $k$-stable,

$$|\{i : x \in A_i\}| \leq k$$

for all $x \in X$. Hence, by (1) and (2),

$$n4^{-k+1} \leq \sum_{i=1}^{n} p_i \leq k$$

which implies $n \leq k4^{k-1}$, and Theorem 1 follows.

Let $\delta(r)$ be the minimal number $n$ such that there exists a function $f : \{0,1\}^n \to R$, where $|R| = r$, which has the following
property

(*) \( f \) depends on all its \( n \) variables, but for every function \( g: R \to S \), where \(|S| < r\), \( g \circ f \) depends on less than \( n \) variables.

For any real number \( \xi \) we let \( \lceil \xi \rceil \) be the least integer not less than \( \xi \).

**Theorem 2.** \( \delta(r) = \binom{r}{2} + \log_2 \binom{r}{2} \).

**Proof.** We put \( s = \binom{r}{2} \) and \( t = \log_2 \binom{r}{2} \). First we show that

\[
\delta(r) \geq s + t.
\]

(This inequality was conjectured by Mycielski and proved first by Ralph McKenzie.) Let \( f \) have the property (*) . Then for every pair \( u, v \in R \), \( u \neq v \) there exists \( 1 \leq i[u,v] \leq n \) such that \( g \circ f \) does not depend on the variable \( x_{i[u,v]} \) whenever \( g(u) = g(v) \). Clearly, if \( u', v' \in R \), \( u' \neq v' \) and \( \{ u', v' \} \neq \{ u, v \} \) then \( i[u', v'] \neq i[u, v] \).

(This already proves that \( \delta(r) \geq s \).) Let \( I = \{ i[u,v]: u, v \in R, u \neq v \} \). Hence

\[
|I| = s.
\]

We need the following lemma.

\((L_2)\) If \( f(x^{i[u,v]}) \neq f(x) \), then, for every \( y \in \{0,1\}^n \) such that \( y_j = x_j \) for \( j \notin I \) and for \( j = i[u,v] \), we have \( f(y) = f(x) \).

To prove this we put \( \tilde{x} = x^{i[u,v]} \). It is enough to check that for all \( j \in I - \{i[u,v]\} \) we have \( f(x^j) = f(x) \); in fact, by symmetry,
the same will then be true about $\sim x$ and hence the point $x^j$ will also satisfy the supposition of $(L_2)$ and $(L_3)$ follows. Then suppose to the contrary that $f(x^j) \neq f(x)$. By our choice of $j$ we have $j = i[u',v']$ for some $u',v' \in R$, $u' \neq v'$, $\{u',v'\} \neq \{u,v\}$.

Thus $f(x) \in \{u',v'\}$ and we can assume without loss of generality that $f(x) = u'$ and $f(x^j) = v' \notin \{u,v\}$. Hence $f(\sim x^j) = v'$ and $f(\sim x^j) \in \{u,v'\}$. But $f(\sim x^j) = f(x) = v \notin \{u,v'\}$. This contradiction completes the proof of $(L_2)$.

Now, by $(L_2)$, for every pair $u,v \in R$, $u \neq v$ there exists an $x \in \{0,1\}^n$ such that $x_i = 0$ for all $i \in I$ and $\{f(x), f(x^{i[u,v]} \} = \{u,v\}$. Then by (4) there are at least $s$ elements $x \in \{0,1\}^n$ with $x_i = 0$ for all $i \in I$. Thus $2^{n-s} \geq s$, i.e., $n \geq s + t$ and (3) follows.

Now we prove the converse inequality

$$\delta(r) \leq s + t$$

It is enough to define some $f: \{0,1\}^R \to R$ with $n = s + t$, $|R| = r$ and the property $(\star)$. Let $P = \{[i,j]: i,j \in \{1,\ldots,r\}, i \neq j\}$.

Thus $|P| = s$. Let $h: P \to \{0,1\}^t$ be one-to-one and $z: P \to \{1,\ldots,s\}$ be one-to-one. For any sequences $x \in \{0,1\}^s$ and $y \in \{0,1\}^t$ we put $xy = (x_1,\ldots,x_s,y_1,\ldots,y_t)$. It is clear that there exists an $f: \{0,1\}^{s+t} \to \{1,\ldots,r\}$ such that $\{f(xh(p)), f(x^z(p)h(p))\} = p$ for all $x \in \{0,1\}^s$ and $p \in P$ and $f(xy) = 1$ if $x \in \{0,1\}^s$ and $y \in \{0,1\}^t$ - range$(h)$. It is easy to check that all such $f$ have the required properties.

$(P_5)$ What are the analogs of Theorem 2 if we restrict $f$'s to be $k$-continuous or $k$-stable functions?
References


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