Some Improved Bounds on the Number of 1-Factors of n-Connected Graphs

by

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1. Introduction

There is an interesting relation between connectivity and the number of distinct 1-factors in a graph. Beineke and Plummer [1] showed that if a graph has a 1-factor and is n-connected, it has n 1-factors. Zaks [3] sharpened the bound on the number of 1-factors to \( n(n-2)(n-4) \cdots 5 \cdot 3 \) for \( n \) odd and \( n(n-2)(n-4) \cdots 4 \cdot 2 \) for \( n \) even. This bound is exact for the complete graphs \( K_{n+1} \) (\( n \) odd) and the party graph \( P_6 \) (\( n=4 \)). We show these are the only graphs that achieve the bound for \( n \geq 3 \), and improve the bound for the remaining cases.

2. Preliminaries

This section gives some definitions and previous results. For terms not defined here, see [2].

A graph \( G \) that has two or more vertices is connected (1-connected) if there is a path between any two vertices. For a positive integer \( n \), \( G \) is \( n \)-connected if, when any \( m \) vertices \( v_i \), \( 0 \leq m < n \), are deleted, the resulting graph \( G - \{v_i| i=1, \ldots, m\} \) is connected. (Note an \( n \)-connected graph has at least \( n+1 \) vertices.) \( G \) has connectivity \( n \) if it is \( n \)-connected but not \((n+1)\)-connected. If \( G \) is connected but \( G - \{v\} \) is not, \( v \) separates \( G \).

A 1-factor (perfect matching) of \( G \) is a subgraph with exactly one edge incident to every vertex of \( G \). A vertex \( v \) is totally covered if every edge \( vw \) incident to \( v \) is in some 1-factor of \( G \). A fundamental result is the following:

**Lemma 1** [3]: A 2-connected graph that has a 1-factor has at least two totally covered vertices.

Define \( F(G) \) as the number of distinct 1-factors of \( G \). For any positive integer \( n \), define \( f(n) \) as the largest integer such that if \( G \)
is an n-connected graph with a 1-factor, then $F(G) \geq f(n)$. Define $g(n)$, a variant of the factorial function, by $g(n) = n$ for $n=1,2$, and $g(n) = ng(n-2)$ for $n \geq 3$. Thus $g(n) = n(n-2) \cdots 5 \cdot 3$ or $n(n-2) \cdots 4 \cdot 2$, depending on whether $n$ is odd or even. An induction using Lemma 1 shows $f(n) \geq g(n)$ for $n \geq 1$.

Define $f^*(n)$ as the second-lowest possible number of 1-factors for an n-connected graph. That is, $f^*(n)$ is the largest integer such that if $G$ is an n-connected graph with $F(G) > f(n)$, then $F(G) \geq f^*(n)$.

The complete graph $K_n$ consists of $n$ vertices and all possible edges between them. If $n$ is odd, $K_{n+1}$ is n-connected and has a 1-factor. In fact, $F(K_{n+1}) = g(n)$; so $f(n) = g(n)$ for $n$ odd.

If $n$ is even, the party graph $P_n$ is $K_n$ with the edges of a 1-factor deleted. $P_n$ is $(n-2)$-connected. It is easy to check $F(P_4) = 2$, $F(P_6) = 8$, and $F(P_{n+2}) = n(F(P_n) + F(P_{n-2}))$ for $n \geq 6$.

3. The New Bounds

We begin by analyzing 3- and 4- connected graphs, confirming conjectures of Zaks on $f^*(3)$ and $f^*(4)$.

**Lemma 2:** $f(3) = 3$; $K_4$ is the only 3-connected graph $G$ with $F(G) = 3$; $f^*(3) = 4$.

**Proof:** Since $F(K_4) = 3$, we have $f(3) \leq 3$. Similarly, $f^*(3) \leq 4$ is shown by the graph consisting of a cycle on six vertices, $v_1v_2v_3v_4v_5v_6$, plus the edges $v_1v_4$, $v_2v_6$, $v_3v_5$. Lemma 1 implies $f(3) \geq 3$, so $f(3) = 3$.

It remains only to show that if $G \neq K_4$ is a 3-connected graph with a 1-factor, then $F(G) \geq 4$. We assume $F(G) = 3$ and derive a contradiction.

$G$ contains a totally covered vertex $t$, by Lemma 1. The assumptions imply $t$ is adjacent to exactly three vertices $v_i$, $i=1,2,3$. Further, each graph $G - \{t, v_i\}$ has exactly one 1-factor, i.e., $F(G - \{t, v_i\}) = 1$. 
Using Lemma 1, we see $G - \{t, v_i\}$ is not 2-connected; however it is 1-connected. Thus there is a vertex $c_i$ separating $G - \{t, v_i\}$ i.e., $G - \{t, v_i, c_i\}$ is not connected. Note $c_i \neq v_j$ for any $i, j$, since $G - \{t, v_i, v_j\}$ is connected. (The assumption $G \neq K_4$ is used here.)

Let $v_i, v_j, v_k$ be the vertices adjacent to $t$, in some order. Graph $G - \{v_i, c_i\}$ is separated by $t$. Thus $v_j$ and $v_k$ are in different components of $G - \{t, v_i, c_i\}$.

The proof is completed by contradicting this fact, i.e., we find a path between $v_j$ and $v_k$ in $G - \{t, v_i, c_i\}$. We do this by finding paths between $v_j$ and $c_k$, $v_k$ and $c_j$, and $c_j$ and $c_k$.

Graph $G - \{t, c_i\}$ is separated by $v_i$. So there is a simple path $P$ between $v_j$ and $v_k$, containing $v_i$. The part of $P$ between $v_j$ and $v_i$ is in $G - \{t, v_k\}$; it contains $c_k$, since $v_j$ and $v_i$ are in different components of $G - \{t, v_k, c_k\}$. This gives the desired path between $v_j$ and $c_k$. Similarly the part of $P$ between $v_k$ and $v_i$ gives the desired path between $v_k$ and $c_j$. Note further, vertices $c_i, c_j$, and $c_k$ are distinct.

Now we find the desired path between $c_j$ and $c_k$. Since $G - \{t, v_i, v_j\}$ is connected, it contains a path $Q$ between $c_j$ and $c_k$, and also a path between $c_j$ and $c_i$. Assume without loss of generality that $Q$ does not contain $c_i$. (If it does, interchange indices $i$ and $j$.) So $Q$ is in $G - \{t, v_i, c_i\}$, and is the last desired path. QED

**Lemma 3:** $f(4) = 8$; $P_6$ is the only 4-connected graph $G$ with $F(G) = 8$; $f^*(4) = 10$.

**Proof:** We see $f(4) \leq 8$ and $f^*(4) \leq 10$ by counting the number of 1-factors in $P_6$ and $P_6 + e$, where the latter graph is $P_6$ plus one extra edge.

Let $G$ be a 4-connected graph with a 1-factor. Let $t$ be a totally covered vertex and let $tv$ be an edge. Then $G - \{t, v\}$ is 2-connected, so $F(G - \{t, v\}) \geq 2$. Since $t$ has degree at least four, we see $F(G) \geq 8$. Thus $f(4) = 8$. 
It remains only to show that $F(G) \geq 10$ if $G \neq P_6$. The argument divides into two cases.

**Case 1:** There is a totally covered vertex $t$ and an edge $tv$, such that $G - \{t, v\}$ is 3-connected.

Apply the results of Lemma 2 to $G - \{t, v\}$. If $G - \{t, v\} \neq K_4$, then $F(G - \{t, v\}) \geq 4$, so $F(G) \geq 4 + 3 \cdot 2 = 10$, as desired. Otherwise if $G - \{t, v\} = K_4$, graph $G$ has six vertices and contains at least the edges of $P_6 + e$. Thus $F(G) \geq F(P_6 + e) = 10$.

**Case 2:** For every totally covered vertex $t$ and every edge $tv$, $G - \{t, v\}$ has connectivity 2.

If a totally covered vertex has degree five or more, then $F(G) \geq 5 \cdot 2 = 10$, as desired. Thus assume all totally covered vertices have degree four. Assume further that $F(G) < 10$. Below we deduce $G = P_6$.

Let $t$ be a totally covered vertex, adjacent to vertices $v_i$, $i = 1, 2, 3, 4$. We first show vertices $v_i$ form a cycle on four vertices, with no other edges joining them.

No three vertices $v_i$ form a cycle. For suppose $v_1v_2v_3$ is a cycle. Then since $G - \{v_4\}$ is 3-connected, $G - \{t, v_4\}$ is 3-connected. But this violates the assumption of Case 2.

Since $F(G) < 10$, assume without loss of generality that $F(G - \{t, v_i\}) = 2$, for $i = 1, 2, 3$. Since $G - \{t, v_i\}$ is 2-connected, it has two totally covered vertices. Both have degree two in $G - \{t, v_i\}$. Any vertex has degree at least four in $G$. So both totally covered vertices of $G - \{t, v_i\}$ are adjacent to $t$ and $v_i$. In particular, both are among the vertices $v_j$.

Thus each vertex $v_i$, $i = 1, 2, 3$, is adjacent to two other vertices $v_j$. Since no three vertices $v_j$ form a cycle, it is easy to see the four vertices $v_j$ form a cycle, with no other edges.
Now we find other totally covered vertices besides $t$. Without loss of generality, assume the cycle found above is $v_1v_2v_3v_4$. Vertex $v_2$ is totally covered in $G - \{t, v_1\}$, for $i=1,3$. Thus $v_2$ is totally covered in $G$. This in turn implies $v_1$ is totally covered in $G$.

The above argument shows the totally covered vertex $v_2$ is adjacent to four vertices joined in a cycle. This cycle is $v_1tv_3s$, where $s$ is a vertex in $G$. Similarly, $v_1$ is adjacent to four vertices joined in a cycle, $v_2tv_4s$. Thus the six vertices $s,t,v_1,v_i,i=1,2,3,4$, and their interconnecting edges, form the party graph $P_6$.

$G$ contains no other vertices, since $G - \{s, v_3, v_4\}$ is connected.

Thus $G = P_6$. QED

Now we extend these results to higher connected graphs. Zaks conjectured that $K_{n+1}$ (n odd) and $P_6$ (n=4) are the only graphs with exactly $g(n)$ 1-factors, for $n \geq 3$. Theorems 4 and 5 confirm this.

**Theorem 4:** Let $n \geq 3$ be odd. Then $f(n) = g(n); K_{n+1}$ is the only $n$-connected graph $G$ with $F(G) = g(n); f^*(n) \geq 4/3 g(n)$.

**Proof:** We show by induction on odd $n$ that if $G \neq K_{n+1}$ is an $n$-connected graph with a 1-factor, then $F(G) \geq 4/3 g(n)$. Lemma 2 establishes the base case, $n=3$. So assume the assertion holds for some $n \geq 3$. Let $G \neq K_{n+3}$ be an $(n+2)$-connected graph with a 1-factor. Let $t$ be a totally covered vertex, and let $tv$ be an edge.

Graph $G - \{t, v\}$ is $n$-connected and has a 1-factor. Further, it is not $K_{n+1}$. For otherwise, each vertex $w$ in $G - \{t, v\}$ is adjacent to both $t$ and $v$, since $w$ has degree at least $n+2$ in $G$. This implies $G = K_{n+3}$, contrary to assumption.

The inductive assertion shows $F(G - \{t, v\}) \geq 4/3 g(n)$. Since there are at least $n+2$ vertices $v$, $F(G) \geq 4/3 g(n+2)$. QED
Theorem 5: Let \( n \geq 6 \) be even. Then \( f(n) \geq \frac{5}{4} g(n) \).

Proof: We first show for \( n=6 \), \( f(6) \geq \frac{5}{4} g(6) = 60 \). Let \( G \) be a 6-connected graph with a 1-factor. It is easy to see a totally covered vertex \( t \) has at least six edges \( tv \) such that \( G - \{t,v\} \neq \mathcal{P}_6 \). So Lemma 3 shows \( F(G) \geq 6 \cdot 10 = 60 \), whence \( f(6) \geq 60 \).

The general case now follows by induction, with \( n=6 \) as the base. QED

Theorem 5 gives an exact bound for \( n=6 \), as shown by the party graph \( P_6 \). We conjecture that for any even \( n \geq 2 \), \( f(n) = F(\mathcal{P}_{n+2}) \).
References

